

# Idempotents of double Burnside algebras, $L$ -enriched bisets, and decomposition of $p$ -biset functors

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**Abstract:** Let  $R$  be a (unital) commutative ring, and  $G$  be a finite group with order invertible in  $R$ . We introduce new idempotents  $\epsilon_{T,S}^G$  in the double Burnside algebra  $RB(G, G)$  of  $G$  over  $R$ , indexed by conjugacy classes of minimal sections  $(T, S)$  of  $G$  (i.e. sections such that  $S \leq \Phi(T)$ ). These idempotents are orthogonal, and their sum is equal to the identity. It follows that for any biset functor  $F$  over  $R$ , the evaluation  $F(G)$  splits as a direct sum of specific  $R$ -modules indexed by minimal sections of  $G$ , up to conjugation.

The restriction of these constructions to the biset category of  $p$ -groups, where  $p$  is a prime number invertible in  $R$ , leads to a decomposition of the category of  $p$ -biset functors over  $R$  as a direct product of categories  $\mathcal{F}_L$  indexed by *atoric*  $p$ -groups  $L$  up to isomorphism.

We next introduce the notions of  *$L$ -enriched biset* and  *$L$ -enriched biset functor* for an arbitrary finite group  $L$ , and show that for an atoric  $p$ -group  $L$ , the category  $\mathcal{F}_L$  is equivalent to the category of  $L$ -enriched biset functors defined over elementary abelian  $p$ -groups.

Finally, the notion of *vertex* of an indecomposable  $p$ -biset functor is introduced (when  $p \in R^\times$ ), and when  $R$  is a field of characteristic different from  $p$ , the objects of the category  $\mathcal{F}_L$  are characterized in terms of vertices of their composition factors.

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## 1. Introduction

Let  $R$  denote throughout a commutative ring (with identity element). For a finite group  $G$ , we consider the double Burnside algebra  $RB(G, G)$  of  $G$  over  $R$ . In the case where the order of  $G$  is invertible in  $R$ , we introduce idempotents  $\epsilon_{T,S}^G$  in  $RB(G, G)$ , indexed by the set  $\mathcal{M}(G)$  of minimal sections of  $G$ , i.e. the set of pairs  $(T, S)$  of subgroups of  $G$  with  $S \trianglelefteq T$  and  $S \leq \Phi(T)$ , where  $\Phi(T)$  is the Frattini subgroup of  $T$  (such sections have been considered in Section 5 of [9]). The idempotent  $\epsilon_{T,S}^G$  only depends of the conjugacy class of  $(T, S)$  in  $G$ . Moreover, the idempotents  $\epsilon_{T,S}^G$ , where  $(T, S)$  runs through a

set  $[\mathcal{M}(G)]$  of representatives of orbits of  $G$  acting on  $\mathcal{M}(G)$  by conjugation, are orthogonal, and their sum is equal to the identity element of  $RB(G, G)$ .

The idempotents  $\epsilon_{G,1}^G$  plays a special role in our construction, and it is denoted by  $\varphi_1^G$ . In particular, when  $F$  is a biset functor over  $R$  (and the order of  $G$  is invertible in  $R$ ), we set  $\delta_\Phi F(G) = \varphi_1^G F(G)$ . We show that  $\delta_\Phi F(G)$  consists of those elements  $u \in F(G)$  such that  $\text{Res}_H^G u = 0$  whenever  $H$  is a proper subgroup of  $G$ , and  $\text{Def}_{G/N}^G u = 0$  whenever  $N$  is a non-trivial normal subgroup of  $G$  contained in  $\Phi(G)$ . This yields moreover a decomposition

$$F(G) \cong \left( \bigoplus_{(T,S) \in [\mathcal{M}(G)]} \delta_\Phi F(T/S) \right)^G \cong \bigoplus_{(T,S) \in [\mathcal{M}(G)]} \delta_\Phi F(T/S)^{N_G(T,S)/T}.$$

Restricting these constructions to the biset category  $RC_p$  of  $p$ -groups with coefficients in  $R$ , where  $p$  is a prime invertible in  $R$ , we get orthogonal idempotents  $b_L$  in the center of  $RC_p$ , indexed by *atoric*  $p$ -groups, i.e. finite  $p$ -groups which cannot be split as a direct product  $C_p \times Q$ , for some  $p$ -group  $Q$ . We show next that every finite  $p$ -group  $P$  admits a unique largest atoric quotient  $P^\oplus$ , well defined up to isomorphism, and that there exists an elementary abelian  $p$ -subgroup  $E$  of  $P$  (non unique in general) such that  $P \cong E \times P^\oplus$ . For a given atoric  $p$ -group  $L$ , we introduce a category  $RC_p^{\#L}$ , defined as a quotient of the subcategory of  $RC_p$  consisting of  $p$ -groups  $P$  such that  $P^\oplus \cong L$ . This leads to a decomposition of the category  $\mathcal{F}_{p,R}$  of  $p$ -biset functors over  $R$  as a direct product

$$\mathcal{F}_{p,R} \cong \prod_{L \in [\mathcal{A}t_p]} \text{Fun}_R(RC_p^{\#L}, R\text{-Mod})$$

of categories of representations of  $RC_p^{\#L}$  over  $R$ , where  $L$  runs through a set  $[\mathcal{A}t_p]$  of isomorphism classes of atoric  $p$ -groups. Similar questions on idempotents in double Burnside algebras and decomposition of biset functors categories have been considered by L. Barker ([1]), R. Boltje and S. Danz ([2], [3]), R. Boltje and B. Külshammer ([4]), and P. Webb ([16]).

In particular, via the above decomposition, to any indecomposable  $p$ -biset functor  $F$  is associated a unique atoric  $p$ -group, called the *vertex* of  $F$ . We show that this vertex is isomorphic to  $Q^\oplus$ , for any  $p$ -group  $Q$  such that  $F(Q) \neq \{0\}$  but  $F$  vanishes on any proper subquotient of  $Q$ .

Going back to arbitrary finite groups, we next introduce the notions of *L-enriched biset* and *L-enriched biset functor*, and show that when  $L$  is an atoric  $p$ -group, the abelian category  $\text{Fun}_R(RC_p^{\#L}, R\text{-Mod})$  is equivalent to the category of  $L$ -enriched biset functors from elementary abelian  $p$ -groups to  $R$ -modules.

The paper is organized as follows: Section 2 is a review of definitions and basic results on Burnside rings and biset functors. Section 3 is concerned

with the algebra  $\mathcal{E}(G)$  obtained by “cutting” the double Burnside algebra  $RB(G, G)$  of a finite group  $G$  by the idempotent  $\widetilde{e}_G^G$  corresponding to the “top” idempotent  $e_G^G$  of the Burnside algebra  $RB(G)$ . Orthogonal idempotents  $\varphi_N^G$  of  $\mathcal{E}(G)$  are introduced, indexed by normal subgroups  $N$  of  $G$  contained in  $\Phi(G)$ . It is shown moreover that if  $G$  is nilpotent, then  $\varphi_1^G$  is central in  $\mathcal{E}(G)$ . In Section 4, the idempotents  $\epsilon_{T,S}^G$  of  $RB(G, G)$  are introduced, leading in Section 5 to the corresponding direct sum decomposition of the evaluation at  $G$  of any biset functor over  $R$ . In Section 6, atoric  $p$ -groups are introduced, and their main properties are stated. In Section 7, the biset category of  $p$ -groups over  $R$  is considered, leading to a splitting of the category  $\mathcal{F}_{p,R}$  of  $p$ -biset functors over  $R$  as a direct product of abelian categories  $\mathcal{F}_L = \text{Fun}_R(R\mathcal{C}_p^{\sharp L}, R\text{-Mod})$  indexed by atoric  $p$ -groups  $L$  up to isomorphism. In Section 8, for an arbitrary finite group  $L$ , the notions of  $L$ -enriched biset and  $L$ -enriched biset functor are introduced, and it is shown that when  $L$  is an atoric  $p$ -group, the category  $\mathcal{F}_L$  is equivalent to the category of  $L$ -enriched biset functors on elementary abelian  $p$ -groups. Finally, in Section 9, for a given atoric  $p$ -group  $L$ , and when  $p$  is invertible in  $R$ , the structure of the category  $\mathcal{F}_L$  is considered, and the notion of vertex of an indecomposable  $p$ -biset functor over  $R$  is introduced. In particular, when  $R$  is a field of characteristic different from  $p$ , it is shown that the objects of  $\mathcal{F}_L$  are those  $p$ -biset functors all composition factors of which have vertex  $L$ .

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## 2. Review of Burnside rings and biset functors

**2.1.** Let  $G$  be a finite group, let  $s_G$  denote the set of subgroups of  $G$ , let  $\overline{s_G}$  denote the set of conjugacy classes of subgroups of  $G$ , and let  $[s_G]$  denote a set of representatives of  $\overline{s_G}$ .

Let  $B(G)$  denote the Burnside ring of  $G$ , i.e. the Grothendieck ring of the category of finite  $G$ -sets. It is a commutative ring, with an identity element, equal to the class of a  $G$ -set of cardinality 1. The additive group  $B(G)$  is a free abelian group on the set  $\{[G/H] \mid H \in [s_G]\}$  of isomorphism classes of transitive  $G$ -sets.

**2.2.** • When  $G$  and  $H$  are finite groups, and  $L$  is a subgroup of  $G \times H$ , set

$$\begin{aligned} p_1(L) &= \{g \in G \mid \exists h \in H, (g, h) \in L\} , \\ p_2(L) &= \{h \in H \mid \exists g \in G, (g, h) \in L\} , \\ k_1(L) &= \{g \in G \mid (g, 1) \in L\} , \\ k_2(L) &= \{h \in H \mid (1, h) \in L\} . \end{aligned}$$

Recall that  $k_i(L) \trianglelefteq p_i(L)$ , for  $i \in \{1, 2\}$ , that  $(k_1(L) \times k_2(L)) \trianglelefteq L$ , and that there are canonical isomorphisms

$$p_1(L)/k_1(L) \cong L/(k_1(L) \times k_2(L)) \cong p_2(L)/k_2(L) .$$

Set moreover  $q(L) = L/(k_1(L) \times k_2(L))$ .

• When  $Z$  is a subgroup of  $G$ , set

$$\Delta(Z) = \{(z, z) \mid z \in Z\} \leq (G \times G) .$$

When  $N$  is a normal subgroup of  $G$ , set

$$\Delta_N(G) = \{(a, b) \in G \times G \mid ab^{-1} \in N\} .$$

It is a subgroup of  $G \times G$ .

• When  $G$ ,  $H$ , and  $K$  are groups, when  $L \leq (G \times H)$  and  $M \leq (H \times K)$ , set

$$L * M = \{(g, k) \in (G \times K) \mid \exists h \in H, (g, h) \in L \text{ and } (h, k) \in M\} .$$

It is a subgroup of  $(G \times K)$ .

**2.3.** When  $G$  and  $H$  are finite groups, a  $(G, H)$ -biset  $U$  is a set endowed with

a left action of  $G$  and a right action of  $H$  which commute. In other words  $U$  is a  $G \times H^{op}$ -set, where  $H^{op}$  is the opposite group of  $H$ . The *opposite biset*  $U^{op}$  is the  $(H, G)$ -biset equal to  $U$  as a set, with actions defined for  $h \in H$ ,  $u \in U$  and  $g \in G$  by  $h \cdot u \cdot g$  (in  $U^{op}$ ) =  $g^{-1}uh^{-1}$  (in  $U$ ).

The Burnside group  $B(G, H)$  is the Grothendieck group of the category of finite  $(G, H)$ -bisets. It is a free abelian group on the set of isomorphism classes  $[(G \times H)/L]$ , for  $L \in [s_{G \times H}]$ , where the  $(G, H)$ -biset structure on  $(G \times H)/L$  is given by

$$\forall a, g \in G, \forall b, h \in H, a \cdot (g, h)L \cdot b = (ag, b^{-1}h)L .$$

When  $G$ ,  $H$ , and  $K$  are finite groups, there is a unique bilinear product

$$\times_H : B(G, H) \times B(H, K) \rightarrow B(G, K)$$

induced by the usual product  $(U, V) \mapsto U \times_H V = (U \times V)/H$  of bisets, where the right action of  $H$  on  $U \times V$  is defined for  $u \in U$ ,  $v \in V$  and  $h \in H$  by  $(u, v) \cdot h = (uh, h^{-1}v)$ . This product will also be denoted as a composition  $(\alpha, \beta) \mapsto \alpha \circ \beta$  or as a product  $(\alpha, \beta) \mapsto \alpha\beta$ .

This leads to the following definitions:

**2.4. Definition:** *The biset category of finite groups  $\mathcal{C}$  is defined as follows:*

- *The objects of  $\mathcal{C}$  are the finite groups.*
- *When  $G$  and  $H$  are finite groups,*

$$\text{Hom}_{\mathcal{C}}(G, H) = B(H, G) .$$

- *When  $G$ ,  $H$ , and  $K$  are finite groups, the composition*

$$\circ : \text{Hom}_{\mathcal{C}}(H, K) \times \text{Hom}_{\mathcal{C}}(G, H) \rightarrow \text{Hom}_{\mathcal{C}}(G, K)$$

*is the product*

$$\times_H : B(K, H) \times B(H, G) \rightarrow B(K, G) .$$

- *The identity morphism of the group  $G$  is the class of the set  $G$ , viewed as a  $(G, G)$ -biset by left and right multiplication.*

*A biset functor is an additive functor from  $\mathcal{C}$  to the category of abelian groups.*

When  $R$  is a commutative (unital) ring, the category  $\mathcal{RC}$  is defined similarly by extending coefficients to  $R$ , i.e. by setting

$$\mathrm{Hom}_{\mathcal{RC}}(G, H) = R \otimes_{\mathbb{Z}} B(H, G) \quad ,$$

which will be simply denoted by  $RB(H, G)$ . A *biset functor over  $R$*  is an  $R$ -linear functor from  $\mathcal{RC}$  to the category  $R\text{-Mod}$  of  $R$ -modules. The category of biset functors over  $R$  (where morphisms are natural transformations of functors) is denoted by  $\mathcal{F}_R$ .

The correspondence sending a  $(G, H)$ -biset  $U$  to its opposite  $U^{op}$  extends to an isomorphism of  $R$ -modules  $RB(G, H) \rightarrow RB(H, G)$ . These isomorphisms give an equivalence of  $R$ -linear categories from  $\mathcal{RC}$  to its opposite category, which is the identity on objects.

**2.5.** Let  $G$  and  $H$  be finite groups, and  $F$  be a biset functor (with values in  $R\text{-Mod}$ ). For any finite  $(H, G)$ -biset  $U$ , the isomorphism class  $[U]$  of  $U$  belongs to  $B(H, G)$ , and it yields an  $R$ -linear map  $F([U]) : F(G) \rightarrow F(H)$ , simply denoted by  $F(U)$ , or even  $f \in F(G) \mapsto U(f) \in F(H)$ . In particular:

- When  $H$  is a subgroup of  $G$ , denote by  $\mathrm{Ind}_H^G$  the set  $G$ , viewed as a  $(G, H)$ -biset for left and right multiplication, and by  $\mathrm{Res}_H^G$  the same set, viewed as an  $(H, G)$ -biset. This gives a map  $\mathrm{Ind}_H^G : F(H) \rightarrow F(G)$ , called induction, and a map  $\mathrm{Res}_H^G : F(G) \rightarrow F(H)$ , called restriction.
- When  $N$  is a normal subgroup of  $G$ , let  $\mathrm{Inf}_{G/N}^G$  denote the set  $G/N$ , viewed as a  $(G, G/N)$ -biset for the left action of  $G$ , and right action of  $G/N$  by multiplication. Also let  $\mathrm{Def}_{G/N}^G$  denote the set  $G/N$ , viewed as a  $(G/N, G)$ -biset. This gives a map  $\mathrm{Inf}_{G/N}^G : F(G/N) \rightarrow F(G)$ , called inflation, and a map  $\mathrm{Def}_{G/N}^G : F(G) \rightarrow F(G/N)$ , called deflation.
- Finally, when  $f : G \rightarrow G'$  is a group isomorphism, denote by  $\mathrm{Iso}(f)$  the set  $G'$ , viewed as a  $(G', G)$ -biset for left multiplication in  $G'$ , and right action of  $G$  given by multiplication by the image under  $f$ . This gives a map  $\mathrm{Iso}(f) : F(G) \rightarrow F(G')$ , called transport by isomorphism.

When  $G$  and  $H$  are finite groups, any  $(G, H)$ -biset is a disjoint union of transitive ones. It follows that any element of  $B(G, H)$  is a linear combination of morphisms of the form  $[(G \times H)/L]$ , where  $L \in s_{G \times H}$ . Moreover, any such morphism factors as

$$(2.6) \quad [(G \times H)/L] = \mathrm{Ind}_{p_1(L)}^G \circ \mathrm{Inf}_{p_1(L)/k_1(L)}^{p_1(L)} \circ \mathrm{Iso}(f) \circ \mathrm{Def}_{p_2(L)/k_2(L)}^{p_2(L)} \circ \mathrm{Res}_{p_2(L)}^H \quad ,$$

where  $f : p_2(L)/k_2(L) \rightarrow p_1(L)/k_1(L)$  is the canonical group isomorphism.

In particular, for  $N \trianglelefteq G$ ,

$$(2.7) \quad [(G \times G)/\Delta_N(G)] = \mathrm{Inf}_{G/N}^G \circ \mathrm{Def}_{G/N}^G \quad .$$

For finite groups  $G, H, K$ , for  $L \leq (G \times H)$  and  $M \leq (H \times K)$ , one has that

$$(2.8) \quad [(G \times H)/L] \times_H [(H \times K)/M] = \sum_{h \in p_2(L) \backslash H / p_1(M)} [(G \times K)/(L * {}^{(h,1)}M)]$$

in  $B(G, K)$ .

**2.9.** When  $G$  is a finite group, the group  $B(G, G)$  is the ring of endomorphisms of  $G$  in the category  $\mathcal{C}$ . This ring is called the double Burnside ring of  $G$ . It is a non-commutative ring (if  $G$  is non trivial), with identity element equal to the class of the set  $G$ , viewed as a  $(G, G)$ -biset for left and right multiplication.

There is a unitary ring homomorphism  $\alpha \mapsto \tilde{\alpha}$  from  $B(G)$  to  $B(G, G)$ , induced by the functor  $X \mapsto \tilde{X}$  from  $G$ -sets to  $(G, G)$ -bisets, where  $\tilde{X} = G \times X$ , with  $(G, G)$ -biset structure given by

$$\forall a, b, g \in G, \forall x \in X, a(g, x)b = (agb, ax) \text{ .}$$

This construction has in particular the following properties ([7], Corollary 2.5.12):

**2.10. Lemma:** *Let  $G$  be a finite group.*

1. *If  $H$  is a subgroup of  $G$ , and  $X$  is a finite  $G$ -set, then there is an isomorphism of  $(G, H)$ -bisets*

$$\tilde{X} \times_G \text{Ind}_H^G \cong \text{Ind}_H^G \times_H \widetilde{\text{Res}_H^G X} \text{ ,}$$

*and an isomorphism of  $(H, G)$ -bisets*

$$\text{Res}_H^G \times_G \tilde{X} \cong \widetilde{\text{Res}_H^G X} \times_H \text{Res}_H^G \text{ .}$$

2. *If  $H$  is a subgroup of  $G$ , and  $Y$  is a finite  $H$ -set, then there is an isomorphism of  $(G, G)$ -bisets*

$$\text{Ind}_H^G \times_H \tilde{Y} \times_H \text{Res}_H^G \cong \widetilde{\text{Ind}_H^G Y} \text{ .}$$

3. *If  $N$  is a normal subgroup of  $G$ , and  $X$  is a finite  $G/N$ -set, then there is an isomorphism of  $(G/N, G)$ -bisets*

$$\tilde{X} \times_{G/N} \text{Def}_{G/N}^G \cong \text{Def}_{G/N}^G \times_G \widetilde{\text{Inf}_{G/N}^G X} \text{ .}$$

4. If  $N$  is a normal subgroup of  $G$ , and  $X$  is a finite  $G$ -set, then there is an isomorphism of  $(G/N, G/N)$ -bisets

$$\text{Def}_{G/N}^G \times_G X \times_G \text{Inf}_{G/N}^G \cong \widetilde{\text{Def}_{G/N}^G X} .$$

**2.11.** Let  $RB(G)$  denote the  $R$ -algebra  $R \otimes_{\mathbb{Z}} B(G)$ . Assume moreover that the order of  $G$  is invertible in  $R$ . Then for  $H \leq G$ , let  $e_H^G \in RB(G)$  be defined by

$$(2.12) \quad e_H^G = \frac{1}{|N_G(H)|} \sum_{K \leq H} |K| \mu(K, H) [G/K] ,$$

where  $\mu$  is the Möbius function of the poset of subgroups of  $G$ . The elements  $e_H^G$ , for  $H \in [s_G]$ , are orthogonal idempotents of  $RB(G)$ , and their sum is equal to the identity element of  $RB(G)$ . It follows that the elements  $\widetilde{e_H^G}$ , for  $H \in [s_G]$ , are orthogonal idempotents of the  $R$ -algebra  $RB(G, G) = R \otimes_{\mathbb{Z}} B(G, G)$ , and the sum of these idempotents is equal to the identity element of  $RB(G, G)$ . The idempotents  $\widetilde{e_H^G}$  play a special role, due to the following lemma:

**2.13. Lemma:** *Let  $R$  be a commutative ring, and  $G$  be a finite group with order invertible in  $R$ .*

1. *Let  $H$  be a proper subgroup of  $G$ . Then*

$$\text{Res}_H^G \circ \widetilde{e_H^G} = 0 \quad \text{and} \quad \widetilde{e_H^G} \circ \text{Ind}_H^G = 0 .$$

2. *When  $N \trianglelefteq G$ , let  $m_{G,N} \in R$  be defined by*

$$m_{G,N} = \frac{1}{|G|} \sum_{\substack{X \in s_G \\ XN=G}} |X| \mu(X, G) .$$

*Then*

$$\text{Def}_{G/N}^G \circ \widetilde{e_H^G} \circ \text{Inf}_{G/N}^G = m_{G,N} \widetilde{e_{G/N}^{G/N}} .$$

3. *Let  $N \trianglelefteq G$ , and suppose that  $N$  is contained in the Frattini subgroup*



$\Phi(G)$  of  $G$ . Then

$$\widetilde{e_{G/N}^{G/N}} \circ \text{Def}_{G/N}^G = \text{Def}_{G/N}^G \circ \widetilde{e_G^G} \quad \text{and} \quad \text{Inf}_{G/N}^G \circ \widetilde{e_{G/N}^{G/N}} = \widetilde{e_G^G} \circ \text{Inf}_{G/N}^G .$$

**Proof :** Assertion 1 follows from Lemma 2.10 and Assertion 1 of Theorem 5.2.4. of [7].

Assertion 2 follows from Lemma 2.10 and Assertion 4 of Theorem 5.2.4. of [7].

Finally the first part of Assertion 3 follows from Lemma 2.10 and Assertion 3 of Theorem 5.2.4. of [7]: indeed  $\text{Inf}_{G/N}^G e_{G/N}^{G/N}$  is equal to the sum of the different idempotents  $e_X^G$  of  $RB(G)$  indexed by subgroups  $X$  such that  $XN = G$ . If  $N \leq \Phi(G)$ , then  $XN = G$  implies  $X\Phi(G) = G$ , hence  $X = G$ . The second part of Assertion 3 follows by taking opposite bisets, since  $\widetilde{e_G^G}$  and  $\widetilde{e_{G/N}^{G/N}}$  are equal to their opposite bisets, and since  $(\text{Def}_{G/N}^G)^{op} \cong \text{Inf}_{G/N}^G$ .  $\square$

**2.14. Remark:** For the same reason, if  $N \leq \Phi(G)$ , then  $m_{G,N} = 1$ .

**2.15. Remark:** It follows from Assertion 1 and Remark 2.6 that if  $G$  and  $H$  are finite groups and if  $L \leq (G \times H)$ , then  $\widetilde{e_G^G}[(G \times H)/L] = 0$  if  $p_1(L) \neq G$ , and  $[(G \times H)/L]\widetilde{e_H^H} = 0$  if  $p_2(L) \neq H$ .

### 3. Idempotents in $\mathcal{E}(G)$

**3.1. Notation:** When  $G$  is a finite group with order invertible in  $R$ , denote by  $\mathcal{E}(G)$  the  $R$ -algebra defined by

$$\mathcal{E}(G) = \widetilde{e_G^G} RB(G, G) \widetilde{e_G^G} .$$

Set

$$\Sigma(G, G) = \{M \in s_{G \times G} \mid p_1(L) = p_2(L) = G\} ,$$

and for  $L \in s_{G \times G}$ , set

$$Y_L = \widetilde{e_G^G} [(G \times G)/L] \widetilde{e_G^G} \in \mathcal{E}(G) .$$

The  $R$ -algebra  $\mathcal{E}(G)$  has been considered in [5], Section 9, in the case  $R$  is a field of characteristic 0. The extension of the results proved there to the

case where  $R$  is a commutative ring in which the order of  $G$  is invertible is straightforward. In particular:

**3.2. Proposition:** *Let  $G$  be a finite group with order invertible in  $R$ .*

1. *If  $L \in s_{G \times G} - \Sigma(G, G)$ , then  $Y_L = 0$ .*
2. *The elements  $Y_L$ , for  $L$  in a set of representatives of  $(G \times G)$ -conjugacy classes on  $\Sigma(G, G)$ , form a  $R$ -basis of  $\mathcal{E}(G)$ .*
3. *For  $L, M \in \Sigma(G, G)$*

$$Y_L Y_M = \frac{m_{G, k_2(L) \cap k_1(M)}}{|G|} \sum_{\substack{Z \leq G \\ Zk_2(L) = Zk_1(M) = G \\ Z \geq k_2(L) \cap k_1(M)}} |Z| \mu(Z, G) Y_{L * \Delta(Z) * M}$$

*in  $\mathcal{E}(G)$ .*

**3.3. Corollary:** *Let  $L, M \in \Sigma(G, G)$ . If one of the groups  $k_2(L)$  or  $k_1(M)$  is contained in  $\Phi(G)$ , then*

$$Y_L Y_M = Y_{L * M} \ .$$

**Proof :** Indeed if  $k_2(L) \leq \Phi(G)$ , then  $Zk_2(L) = G$  implies  $Z\Phi(G) = G$ , hence  $Z = G$ . Similarly, if  $k_1(M) \leq \Phi(G)$ , then  $Zk_1(M) = G$  implies  $Z = G$ . In each case, Proposition 3.2 then gives

$$Y_L Y_M = m_{G, k_2(L) \cap k_1(M)} Y_{L * M} \ ,$$

and moreover  $m_{G, k_2(L) \cap k_1(M)} = 1$  since  $k_2(L) \cap k_1(M) \leq \Phi(G)$ , by Remark 2.14.  $\square$

**3.4. Notation:** *For a normal subgroup  $N$  of  $G$  such that  $N \leq \Phi(G)$ , set*

$$\varphi_N^G = \sum_{\substack{M \trianglelefteq G \\ N \leq M \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, M) Y_{\Delta_M(G)} \ ,$$

*where  $\mu_{\trianglelefteq G}$  is the Möbius function of the poset of normal subgroups of  $G$ .*

**3.5. Proposition:** *Let  $N \trianglelefteq G$  with  $N \leq \Phi(G)$ . Then*

$$\varphi_N^G = \text{Inf}_{G/N}^G \varphi_1^{G/N} \text{Def}_{G/N}^G .$$

**Proof :** Indeed if  $N \leq M \trianglelefteq G$ , then  $\mu_{\trianglelefteq G}(N, M) = \mu_{\trianglelefteq G/N}(\mathbf{1}, M/N)$ . Since moreover  $N \leq \Phi(G)$ , setting  $\bar{G} = G/N$  and  $\bar{M} = M/N$ , we have by Lemma 2.13

$$\begin{aligned} \text{Inf}_{G/N}^G Y_{\Delta_{\bar{G}}(\bar{M})} \text{Def}_{G/N}^G &= \text{Inf}_{G/N}^G \circ \widetilde{e}_{\bar{G}}((\bar{G} \times \bar{G})/\Delta_{\bar{G}}(\bar{M})) \widetilde{e}_{\bar{G}} \circ \text{Def}_{G/N}^G \\ &= \widetilde{e}_{\bar{G}} \circ \text{Inf}_{G/N}^G((\bar{G} \times \bar{G})/\Delta_{\bar{G}}(\bar{M})) \text{Def}_{G/N}^G \circ \widetilde{e}_{\bar{G}} \\ &= \widetilde{e}_{\bar{G}}((G \times G)/\Delta_M(G)) \widetilde{e}_{\bar{G}} \\ &= Y_{\Delta_M(G)} , \end{aligned}$$

since  $\text{Inf}_{G/N}^G((\bar{G} \times \bar{G})/\Delta_{\bar{G}}(\bar{M})) \text{Def}_{G/N}^G = (G \times G)/\Delta_M(G)$ .  $\square$

**3.6. Proposition:**

1. *Let  $N \trianglelefteq G$ , with  $N \leq \Phi(G)$ . Then*

$$\begin{aligned} \varphi_N^G &= \widetilde{e}_{\bar{G}} \times_G \left( \sum_{\substack{M \trianglelefteq G \\ N \leq M \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, M)[(G \times G)/\Delta_M(G)] \right) \\ &= \left( \sum_{\substack{M \trianglelefteq G \\ N \leq M \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, M)[(G \times G)/\Delta_M(G)] \right) \times_G \widetilde{e}_{\bar{G}} . \end{aligned}$$

2. *In particular*

$$\varphi_1^G = \frac{1}{|G|} \sum_{\substack{X \leq G, M \trianglelefteq G \\ M \leq \Phi(G) \leq X \leq G}} |X| \mu(X, G) \mu_{\trianglelefteq G}(\mathbf{1}, M) \text{Indinf}_{X/M}^G \circ \text{Defres}_{X/M}^G .$$

**Proof :** For Assertion 1, by definition

$$\varphi_N^G = \sum_{\substack{M \trianglelefteq G \\ N \leq M \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, M) \widetilde{e}_{\bar{G}}[(G \times G)/\Delta_M(G)] \times_G \sum_{X \leq G} \frac{|X|}{|G|} \mu(X, G)[(G \times G)/\Delta(X)] .$$

Moreover  $[(G \times G)/\Delta_M(G)] \times_G [(G \times G)/\Delta(X)] = [(G \times G)/(\Delta_M(G) * \Delta(X))]$ , by (2.8), and  $\Delta_M(G) * \Delta(X) = \{(xm, x) \mid x \in X, m \in M\}$ . The first

projection of this group is equal to  $XM$ , hence it is equal to  $G$  if and only if  $X = G$ , since  $M \leq \Phi(G)$ . The first equality of Assertion 1 follows, by Remark 2.15. The second one follows by taking opposite bisets, since  $\widetilde{e}_G^G$  and  $[(G \times G)/\Delta_M(G)]$  are equal to their opposite.

Assertion 2 follows in the special case where  $N = 1$ , expanding  $\widetilde{e}_G^G$  as

$$\widetilde{e}_G^G = \frac{1}{|G|} \sum_{X \leq G} |X| \mu(X, G) [(G \times G)/\Delta(X)] ,$$

observing that  $\mu(X, G) = 0$  unless  $X \geq \Phi(G)$ , and that if  $X \geq \Phi(G) \geq M$ , then

$$[(G \times G)/\Delta(X)] \circ [(G \times G)/\Delta_M(G)] = [(G \times G)/\Delta_M(X)] ,$$

which is equal to  $\text{Indinf}_{X/M}^G \circ \text{Defres}_{X/M}^G$ .  $\square$

### 3.7. Corollary:

1. Let  $H < G$ . Then  $\text{Res}_H^G \varphi_N^G = 0$  and  $\varphi_N^G \text{Ind}_H^G = 0$ .
2. Let  $M \trianglelefteq G$ . If  $M \cap \Phi(G) \not\leq N$ , then  $\text{Def}_{G/M}^G \varphi_N^G = 0$  and  $\varphi_N^G \text{Inf}_{G/M}^G = 0$ .

**Proof :** The first part of Assertion 1 follows from Lemma 2.13, since

$$\text{Res}_H^G \varphi_N^G = \text{Res}_H^G \widetilde{e}_G^G \varphi_N^G = 0 .$$

The second part follows by taking opposite bisets.

For Assertion 2, let  $P = M \cap \Phi(G)$ . Since  $\text{Def}_{G/M}^G = \text{Def}_{G/M}^{G/P} \circ \text{Def}_{G/P}^G$ , it suffices to consider the case  $M = P$ , i.e. the case where  $M \leq \Phi(G)$ . Then, since  $[(G \times G)/\Delta_M(G)] = \text{Inf}_{G/M}^G \text{Def}_{G/M}^G$  for any  $M \trianglelefteq G$ , by 2.7, and since  $\text{Def}_{G/M}^G \text{Inf}_{G/Q}^G = \text{Inf}_{G/MQ}^{G/M} \text{Def}_{G/MQ}^{G/Q}$  for any  $M, Q \trianglelefteq G$ ,

$$\begin{aligned} \text{Def}_{G/M}^G \varphi_N^G &= \text{Def}_{G/M}^G \sum_{\substack{Q \trianglelefteq G \\ N \leq Q \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, Q) \text{Inf}_{G/Q}^G \text{Def}_{G/Q}^G \widetilde{e}_G^G \\ &= \sum_{\substack{Q \trianglelefteq G \\ N \leq Q \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, Q) \text{Inf}_{G/MQ}^G \text{Def}_{G/MQ}^G \widetilde{e}_G^G \\ &= \sum_{\substack{P \trianglelefteq G \\ N \leq P \leq \Phi(G)}} \left( \sum_{\substack{Q \trianglelefteq G \\ N \leq Q \leq \Phi(G) \\ QM=P}} \mu_{\trianglelefteq G}(N, Q) \right) \text{Inf}_{G/P}^G \text{Def}_{G/P}^G \widetilde{e}_G^G . \end{aligned}$$

Now for a given  $P \trianglelefteq G$  with  $P \subseteq \Phi(G)$ , the sum  $\sum_{\substack{Q \trianglelefteq G \\ N \leq Q \leq \Phi(G) \\ QM=P}} \mu_{\trianglelefteq G}(N, Q)$  is

equal to zero unless  $NM = N$ , that is  $M \leq N$ , by classical properties of the Möbius function ([15] Corollary 3.9.3). This proves the first part of Assertion 2, and the second part follows by taking opposite bisets.  $\square$

**3.8. Theorem:** *Let  $G$  be a finite group with order invertible in  $R$ .*

1. *The elements  $\varphi_N^G$ , for  $N \trianglelefteq G$  with  $N \leq \Phi(G)$ , form a set of orthogonal idempotents in the algebra  $\mathcal{E}(G)$ , and their sum is equal to the identity element  $\widetilde{e}_G^G$  of  $\mathcal{E}(G)$ .*
2. *Let  $N \trianglelefteq G$  with  $N \leq \Phi(G)$ , and let  $H$  be a finite group.*
  - (a) *If  $L \leq (G \times H)$ , then  $\varphi_N^G \times_G [(G \times H)/L] = 0$  unless  $p_1(L) = G$  and  $k_1(L) \cap \Phi(G) \leq N$ .*
  - (b) *If  $L' \leq (H \times G)$ , then  $[(H \times G)/L'] \times_G \varphi_N^G = 0$  unless  $p_2(L') = G$  and  $k_2(L') \cap \Phi(G) \leq N$ .*

**Proof :** For  $N \trianglelefteq G$ , set  $u_N^G = Y_{\Delta_N(G)}$ . Since  $\Delta_N(G) * \Delta_M(G) = \Delta_{NM}(G)$  for any normal subgroups  $N$  and  $M$  of  $G$ , it follows from Corollary 3.3 that if either  $N$  or  $M$  is contained in  $\Phi(G)$ , then  $u_N^G u_M^G = u_{NM}^G$ .

Now Assertion 1 follows from the following general procedure for building orthogonal idempotents (see [13] Theorem 10.1 for details): we have a finite lattice  $P$  (here  $P$  is the lattice of normal subgroups of  $G$  contained in  $\Phi(G)$ ), and a set of elements  $g_x$  of a ring  $A$ , for  $x \in P$  (here  $A = \mathcal{E}(G)$  and  $g_N = u_N^G$ ), with the property that  $g_x g_y = g_{x \vee y}$  for any  $x, y \in P$ , and  $g_0 = 1$ , where 0 is the smallest element of  $P$  (here this element is the trivial subgroup of  $G$ , and  $u_1^G = Y_{\Delta_1(G)} = \widetilde{e}_G^G$ ). The elements  $f_x$  defined for  $x \in P$  by

$$f_x = \sum_{\substack{y \in P \\ x \leq y}} \mu(x, y) g_y ,$$

where  $\mu$  is the Möbius function of  $P$ , are orthogonal idempotents of  $A$ , and their sum is equal to the identity element of  $A$ . This is exactly Assertion 1 (since  $f_x = \varphi_N^G$  here, for  $x = N \in P$ ).

Let  $L \leq (G \times H)$ , then by 2.6

$$\varphi_N^G \times_G [(G \times H)/L] = \varphi_N^G \circ \text{Ind}_{p_1(L)}^G \circ [(p_1(L) \times H)/L] = 0$$

unless  $p_1(L) = G$ , by Corollary 3.7. And if  $p_1(L) = G$ , then by 2.6

$$\varphi_N^G \times_G [(G \times H)/L] = \varphi_N^G \circ \text{Inf}_{G/k_1(L)}^G \circ [(G/k_1(L) \times H)/L_1] ,$$

for some subgroup  $L_1$  of  $(G/k_1(L) \times H)$ . Again, by Corollary 3.7 this is equal to 0 unless  $k_1(L) \cap \Phi(G) \leq N$ . The proof of Assertion (b) is similar. Alternatively, one can take opposite bisets in (a).  $\square$

**3.9. Proposition:** *Let  $G$  be a finite group with order invertible in  $R$ .*

1. *Let  $L \in \Sigma(G, G)$ . Then*

$$\varphi_1^G Y_L = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{(N \times \mathbf{1})L} .$$

*This is non zero if and only if  $k_1(L) \cap \Phi(G) = \mathbf{1}$ . Similarly*

$$Y_L \varphi_1^G = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{L(\mathbf{1} \times N)} ,$$

*and  $Y_L \varphi_1^G \neq 0$  if and only if  $k_2(L) \cap \Phi(G) = \mathbf{1}$ .*

2. *The elements  $\varphi_1^G Y_L$  (resp.  $Y_L \varphi_1^G$ ), when  $L$  runs through a set of representatives of conjugacy classes of elements of  $\Sigma(G, G)$  such that  $k_1(L) \cap \Phi(G) = \mathbf{1}$  (resp  $k_2(L) \cap \Phi(G) = \mathbf{1}$ ), form an  $R$ -basis of the right ideal  $\varphi_1^G \mathcal{E}(G)$  (resp. the left ideal  $\mathcal{E}(G) \varphi_1^G$ ) of  $\mathcal{E}(G)$ .*

**Proof :** Let  $L \in \Sigma(G, G)$ . By Proposition 3.8, we have

$$\begin{aligned} \varphi_1^G Y_L &= \widetilde{e}_G^G \times_G \left( \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) [(G \times G) / \Delta_N(G)] \right) \times_G [(G \times G) / L] \times_G \widetilde{e}_G^G \\ &= \widetilde{e}_G^G \times_G \left( \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) [(G \times G) / \Delta_N(G) * L] \right) \times_G \widetilde{e}_G^G \\ &= \widetilde{e}_G^G \times_G \left( \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) [(G \times G) / (N \times \mathbf{1})L] \right) \times_G \widetilde{e}_G^G . \\ &= \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{(N \times \mathbf{1})L} . \end{aligned}$$

Set  $M = k_1(L) \cap \Phi(G)$ . Then  $M \trianglelefteq G$ , and  $(N \times \mathbf{1})L = (NM \times \mathbf{1})L$  for any

normal subgroup  $N$  of  $G$  contained in  $\varphi(G)$ . Thus

$$(3.10) \quad \varphi_1^G Y_L = \sum_{\substack{P \trianglelefteq G \\ M \leq P \leq \Phi(G)}} \left( \sum_{\substack{N \trianglelefteq G \\ NM=P}} \mu_{\trianglelefteq G}(\mathbf{1}, N) \right) Y_{(P \times \mathbf{1})L} .$$

If  $M \neq \mathbf{1}$ , then  $\left( \sum_{\substack{N \trianglelefteq G \\ NM=P}} \mu_{\trianglelefteq G}(\mathbf{1}, N) \right) = 0$  for any  $P \trianglelefteq G$  with  $M \leq P \leq \Phi(G)$ .

Hence  $\varphi_1^G Y_L = 0$  in this case. And if  $M = \mathbf{1}$ , Equation (3.10) reads

$$\varphi_1^G Y_L = \sum_{\substack{P \trianglelefteq G \\ P \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, P) Y_{(P \times \mathbf{1})L} .$$

The element  $Y_{(P \times \mathbf{1})L}$  is equal to  $Y_L$  if and only if  $(P \times \mathbf{1})L$  is conjugate to  $L$ . This implies that  $k_1((P \times \mathbf{1})L)$  is conjugate to (hence equal to)  $k_1(L)$ . Thus  $P \leq k_1((P \times \mathbf{1})L) \leq k_1(L) \cap \Phi(G)$ , hence  $P = \mathbf{1}$ . So the coefficient of  $Y_L$  in  $\varphi_1^G Y_L$  is equal to 1, hence  $\varphi_1^G Y_L \neq 0$ . The remaining statements of Assertion 1 follow by taking opposite bisets.

Assertion 2 follows from Proposition 3.2, and from the fact that the coefficient of  $Y_L$  in  $\varphi_1^G Y_L$  is equal to 1 when  $k_1(L) \cap \Phi(G) = \mathbf{1}$ .  $\square$

**3.11. Corollary:** *Let  $G$  be a finite group of order invertible in  $R$ . If every minimal (non-trivial) normal subgroup of  $G$  is contained in  $\Phi(G)$ , then  $\varphi_1^G$  is central in  $\mathcal{E}(G)$ , and the algebra  $\varphi_1^G \mathcal{E}(G)$  is isomorphic to  $R\text{Out}(G)$ .*

**Proof :** Indeed if  $L \in \Sigma(L, L)$  and  $\varphi_1^G Y_L \neq 0$ , then  $k_1(L) \cap \Phi(G) = \mathbf{1}$ . It follows that  $k_1(L)$  contains no minimal normal subgroup of  $G$ , and then  $k_1(L) = \mathbf{1}$ . Equivalently  $q(L) \cong p_1(L)/k_1(L) \cong G \cong p_2(L)/k_2(L)$ , i.e.  $k_2(L) = G$  also, or equivalently  $k_2(L) \cap \Phi(G) = \mathbf{1}$ . Hence  $\varphi_1^G Y_L \neq 0$  if and only if  $Y_L \varphi_1^G \neq 0$ , and in this case, there exists an automorphism  $\theta$  of  $G$  such that

$$L = \Delta_\theta(G) = \{(\theta(x), x) \mid x \in G\} .$$

In this case for any normal subgroup  $N$  of  $G$  contained in  $\Phi(G)$

$$\begin{aligned} (N \times \mathbf{1})L &= \{(a, b) \in G \times G \mid a\theta(b)^{-1} \in N\} \\ &= \{(a, b) \in G \times G \mid a^{-1}\theta(b) \in N\} \\ &= L(\mathbf{1} \times \theta^{-1}(N)) . \end{aligned}$$

Now  $N \mapsto \theta^{-1}(N)$  is a permutation of the set of normal subgroups of  $G$  contained in  $\Phi(G)$ . Moreover  $\mu_{\trianglelefteq G}(\mathbf{1}, N) = \mu_{\trianglelefteq G}(\mathbf{1}, \theta^{-1}(N))$ .

It follows that  $\varphi_1^G Y_L = Y_L \varphi_1^G$ , so  $\varphi_1^G$  is central in  $\mathcal{E}(G)$ . Moreover the map  $\theta \in \text{Aut}(G) \mapsto \varphi_1^G Y_{\Delta_\theta(G)}$  clearly induces an algebra isomorphism  $R\text{Out}(G) \rightarrow \varphi_1^G \mathcal{E}(G)$ .  $\square$

**3.12. Theorem:** *Let  $G$  be a finite group with order invertible in  $R$ . If  $G$  is nilpotent, then  $\varphi_1^G$  is a central idempotent of  $\mathcal{E}(G)$ .*

**Proof :** Let  $L \in \Sigma(G, G)$ . Setting  $Q = q(L)$ , there are two surjective group homomorphisms  $s, t : G \rightarrow Q$  such that  $L = \{(x, y) \in G \times G \mid s(x) = t(y)\}$ . Then  $k_1(L) = \text{Ker } s$  and  $k_2(L) = \text{Ker } t$ . Now by Proposition 3.9

$$\varphi_1^G Y_L = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{(N \times \mathbf{1})L} ,$$

and this is non zero if and only if  $\text{Ker } s \cap \Phi(G) = \mathbf{1}$ . Now  $s(\Phi(G))$  is equal to  $\Phi(Q)$  since  $G$  is nilpotent: indeed  $G = \prod_p G_p$  (resp.  $Q = \prod_p Q_p$ ) is the direct product of its  $p$ -Sylow subgroups  $G_p$  (resp.  $Q_p$ ), and  $s$  induces a surjective group homomorphism  $G_p \rightarrow Q_p$ , for any prime  $p$ . Moreover  $\Phi(G) = \prod_p \Phi(G_p)$  (resp.  $\Phi(Q) = \prod_p \Phi(Q_p)$ ). Finally  $\Phi(G_p)$  is the subgroup of  $G_p$  generated by commutators and  $p$ -powers of elements of  $G_p$ , hence it maps by  $s$  onto the subgroup of  $Q_p$  generated by commutators and  $p$ -powers of elements of  $Q_p$ , that is  $\Phi(Q_p)$ . Similarly  $t(\Phi(G)) = \Phi(Q)$ .

If  $\text{Ker } s \cap \Phi(G) = \mathbf{1}$ , it follows that  $s$  induces an isomorphism from  $\Phi(G)$  to  $\Phi(Q)$ . Then the surjective homomorphism  $\Phi(G) \rightarrow \Phi(Q)$  induced by  $t$  is also an isomorphism, and in particular  $\text{Ker } t \cap \Phi(G) = \mathbf{1}$ .

Let  $D = L \cap (\Phi(G) \times \Phi(G))$ . Then  $k_1(D) \subseteq k_1(L) \cap \Phi(G) = \text{Ker } s \cap \Phi(G)$ , hence  $k_1(D) = \mathbf{1}$ . Similarly  $k_2(L) \subseteq k_2(L) \cap \Phi(G) = \text{Ker } t \cap \Phi(G) = \mathbf{1}$ , hence  $k_2(D) = \mathbf{1}$ . Since  $s(\Phi(G)) = \Phi(Q) = t(\Phi(G))$ , we have moreover  $p_1(D) = \Phi(G) = p_2(D)$ . It follows that there is an automorphism  $\alpha$  of  $\Phi(G)$  such that  $D = \{(x, \alpha(x)) \mid x \in \Phi(G)\}$ .

Moreover for any  $y \in G$ , there exists  $z \in G$  such that  $(y, z) \in L$ . It follows that  $(x^y, \alpha(x)^z) \in D$  for any  $x \in \Phi(G)$ , that is  $\alpha(x^y) = \alpha(x)^z$ . In particular if  $N$  is a normal subgroup of  $G$  contained in  $\Phi(G)$ , then so is  $\alpha(N)$ . Hence  $\alpha$  induces an automorphism of the poset of normal subgroups of  $G$  contained in  $\Phi(G)$ . In particular  $\mu_{\trianglelefteq G}(\mathbf{1}, N) = \mu_{\trianglelefteq G}(\mathbf{1}, \alpha(N))$ .

Moreover for  $n \in N$  and  $(y, z) \in L$ , we have

$$(n, 1)(y, z) = (y, z)(n^y, 1) = (y, z)(n^y, \alpha(n^y))(1, \alpha(n^y)^{-1}) .$$

Since  $(n^y, \alpha(n^y)) \in D \leq L$ , we have  $(N \times \mathbf{1})L = L(\mathbf{1} \times \alpha(N))$ . It follows



that

$$\begin{aligned}
\varphi_1^G Y_L &= \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{(N \times \mathbf{1})L} = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{L(\mathbf{1} \times \alpha(N))} \\
&= \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, \alpha(N)) Y_{L(\mathbf{1} \times \alpha(N))} = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{L(\mathbf{1} \times N)} \\
&= Y_L \varphi_1^G,
\end{aligned}$$

as was to be shown.  $\square$

**3.13. Remark:** When  $G$  is not nilpotent, it is not true in general that  $\varphi_1^G$  is central in  $\mathcal{E}(G)$ . This is because  $t(\Phi(G))$  need not be equal to  $\Phi(Q)$  for a surjective group homomorphism  $t : G \rightarrow Q$ . For example, there is a surjection  $t$  from the group  $G = C_4 \times (C_5 \rtimes C_4)$  to  $Q = C_4$  with kernel  $C_4 \times C_5$  containing  $\Phi(G) = C_2 \times \mathbf{1}$ , and another surjection  $s : G \rightarrow Q$  with kernel  $\mathbf{1} \times (C_5 \rtimes C_4)$  intersecting trivially  $\Phi(G)$ . In this case, the group  $L = \{(x, y) \in G \times G \mid s(x) = t(y)\}$  is in  $\Sigma(G, G)$ , and  $k_1(L) \cap \Phi(G) = \mathbf{1}$ , but  $k_2(L) \cap \Phi(G) = \Phi(G) \neq \mathbf{1}$ . By Proposition 3.9, we have  $\varphi_1^G Y_L \neq 0$  and  $Y_L \varphi_1^G = 0$ , so  $\varphi_1^G$  is not central in  $\mathcal{E}(G)$ .

## 4. Idempotents in $RB(G, G)$

**4.1. Definition:** When  $G$  is a finite group, a section  $(T, S)$  of  $G$  is a pair of subgroups of  $G$  such that  $S \trianglelefteq T$ .

A section  $(T, S)$  is called minimal (cf. [9]) if  $S \leq \Phi(T)$ . Let  $\mathcal{M}(G)$  denote the set of minimal sections of  $G$ .

A group  $H$  is called a subquotient of  $G$  (notation  $H \sqsubseteq G$ ) if there exists a section  $(T, S)$  of  $G$  such that  $T/S \cong H$ .

A section  $(T, S)$  is minimal if and only if the only subgroup  $H$  of  $T$  such that  $H/(H \cap S) \cong T/S$  is  $T$  itself.

**4.2. Notation:** Let  $G$  be a finite group, and let  $(T, S)$  be a section of  $G$ .

1. Let  $\text{Indinf}_{T/S}^G \in B(G, T/S)$  denote (the isomorphism class of) the  $(G, T/S)$ -biset  $G/S$ , and let  $\text{Defres}_{T/S}^G \in B(T/S, G)$  denote (the isomorphism class of) the  $(T/S, G)$ -biset  $S \backslash G$ .
2. Let  $R$  be a commutative ring in which the order of  $G$  is invertible. Let

$u_{T,S}^G \in RB(G, T/S)$  be defined by

$$u_{T,S}^G = \text{Indinf}_{T/S}^G \varphi_1^{T/S} ,$$

and let  $v_{T,S}^G \in RB(T/S, G)$  be defined by

$$v_{T,S}^G = \varphi_1^{T/S} \text{Defres}_{T/S}^G .$$

**4.3. Remark:** Observe that  $v_{T,S}^G = (u_{T,S}^G)^{op}$ : indeed  $(G/S)^{op} \cong S \backslash G$ , and  $(\varphi_1^{T/S})^{op} = \varphi_1^{T/S}$ .

**4.4. Theorem:** Let  $G$  be a finite group with order invertible in  $R$ .

1. If  $(T, S)$  and  $(T', S')$  are minimal sections of  $G$ , then

$$v_{T',S'}^G u_{T,S}^G = 0$$

unless  $(T, S)$  and  $(T', S')$  are conjugate in  $G$ .

2. If  $(T, S)$  is a minimal section of  $G$ , then

$$v_{T,S}^G u_{T,S}^G = \varphi_1^{T/S} \left( \sum_{g \in N_G(T,S)/T} \text{Iso}(c_g) \right) ,$$

where  $N_G(T, S) = N_G(T) \cap N_G(S)$ , and  $c_g$  is the automorphism of  $T/S$  induced by conjugation by  $g$ .

**Proof :** Indeed  $(S' \backslash G) \times_G (G/S) \cong S' \backslash G/S$  as a  $(T'/S', T/S)$ -biset. Hence

$$v_{T',S'}^G u_{T,S}^G = \varphi_1^{T'/S'} \left( \sum_{g \in T' \backslash G/T} S' \backslash T' g T/S \right) \varphi_1^{T/S} .$$

For any  $g \in G$ , the  $(T'/S', T/S)$ -biset  $S' \backslash T' g T/S$  is transitive, isomorphic to  $((T'/S') \times (T/S))/L_g$ , where

$$L_g = \{(t' S', tS) \in (T'/S') \times (T/S) \mid t' g t^{-1} \in S' g S\} .$$

Then  $t' S' \in p_1(L_g)$  if and only if  $t' \in S' \cdot g T g^{-1} \cap T'$ . Hence

$$p_1(L_g) = ({}^g T \cap T') S' / S' .$$

Similarly  $p_2(L_g) = (T'^g \cap T) S / S$ . In particular  $p_1(L_g) = T'/S'$  if and only if  $({}^g T \cap T') S' = T'$ , i.e.  ${}^g T \cap T' = T'$ , since  $S' \leq \Phi(T')$ . Thus  $p_1(L_g) = T'/S'$

if and only if  $T' \leq {}^g T$ . Similarly  $p_2(L_g) = T/S$  if and only if  $T \leq T'^g$ . By Theorem 3.8, it follows that  $\varphi_1^{T'/S'}(S' \setminus T'gT/S)\varphi_1^{T/S} = 0$  unless  $T' = {}^g T$ .

Assume now that  $T' = {}^g T$ . Then  $t'S' \in k_1(L_g)$  if and only if  $t'$  lies in  $S' \cdot gSg^{-1} \cap T'$ . Hence

$$k_1(L_g) = ({}^g S \cap T')S'/S' ,$$

and similarly  $k_2(L_g) = (S'^g \cap T)S/S$ . But since  $S \leq \Phi(T)$  and  $S \trianglelefteq T$ , it follows that  ${}^g S \trianglelefteq {}^g T = T'$  and  ${}^g S \leq {}^g \Phi(T) = \Phi(T')$ . Hence  ${}^g S \cdot S'/S'$  is contained in  $k_1(L_g) \cap \Phi(T')/S'$ . Moreover  $\Phi(T')/S' = \Phi(T'/S')$ , as

$$\Phi(T'/S') = \bigcap_{S' \leq M' < T'} (M'/S') = \bigcap_{M' < T'} (M'/S') = (\bigcap_{M' < T'} M')/S' = \Phi(T')/S' ,$$

where  $M'$  runs through maximal subgroups of  $T'$ , which all contain  $S'$  since  $S' \leq \Phi(T')$ .

It follows that if  $k_1(L_g) \cap \Phi(T'/S') = \mathbf{1}$ , then  ${}^g S \cdot S' = S'$ , that is  ${}^g S \leq S'$ . Similarly if  $k_2(L_g) \cap \Phi(T/S) = \mathbf{1}$ , then  $S'^g \leq S$ . By Theorem 3.8, it follows that  $\varphi_1^{T'/S'}(S' \setminus T'gT/S)\varphi_1^{T/S} = 0$  unless  $T' = {}^g T$  and  $S' = {}^g S$ . This proves Assertion 1.

For Assertion 2, the same computation shows that

$$v_{T,S}^G u_{T,S}^G = \sum_{g \in N_G(T,S)/T} \varphi_1^{T/S}(S \setminus TgT/S)\varphi_1^{T/S} .$$

But  $S \setminus TgT/S = gT/S$  if  $g \in N_G(T, S)$ , and this  $(T/S, T/S)$ -biset is isomorphic to  $\text{Iso}(c_g)$ . Assertion 2 follows, since moreover  $\varphi_1^{T/S}$  commutes with any biset of the form  $\text{Iso}(\theta)$ , where  $\theta$  is an automorphism of  $T/S$ .  $\square$

**4.5. Notation:** For a minimal section  $(T, S)$  of the group  $G$ , set

$$\epsilon_{T,S}^G = \frac{1}{|N_G(T,S):T|} u_{T,S}^G v_{T,S}^G = \frac{1}{|N_G(T,S):T|} \text{Indinf}_{T/S}^G \varphi_1^G \text{Defres}_{T/S}^G \in RB(G, G) .$$

Note that  $\epsilon_{T,S}^G = \epsilon_{gT, {}^g S}^G$  for any  $g \in G$ , and that  $\epsilon_{G,N}^G = \varphi_N^G$  when  $N \trianglelefteq G$  and  $N \leq \Phi(G)$ , by Proposition 3.5.

**4.6. Proposition:** Let  $(T, S)$  be a minimal section of  $G$ . Then

$$\epsilon_{T,S}^G = \frac{1}{|N_G(T, S)|} \sum_{\substack{X \leq T, M \trianglelefteq T \\ S \leq M \leq \Phi(T) \leq X \leq T}} |X| \mu(X, T) \mu_{\trianglelefteq T}(S, M) \text{Indinf}_{X/M}^G \circ \text{Defres}_{X/M}^G .$$

**Proof :** This is a straightforward consequence of the above definition of  $\epsilon_{T,S}^G$ , and from Assertion 2 of Proposition 3.6.  $\square$

**4.7. Theorem:** *Let  $G$  be a finite group with order invertible in  $R$ , let  $[\mathcal{M}(G)]$  be a set of representatives of conjugacy classes of minimal sections of  $G$ . Then the elements  $\epsilon_{T,S}^G$ , for  $(T,S) \in [\mathcal{M}(G)]$ , are orthogonal idempotents of  $RB(G,G)$ , and their sum is equal to the identity element of  $RB(G,G)$ .*

**Proof :** Let  $(T,S)$  and  $(T',S')$  be distinct elements of  $[\mathcal{M}(G)]$ . Then

$$\epsilon_{T',S'}^G \epsilon_{T,S}^G = \frac{1}{|N_G(T',S'):T'|} \frac{1}{|N_G(T,S):T|} u_{T',S'}^G v_{T',S'}^G u_{T,S}^G v_{T,S}^G = 0 \quad ,$$

since  $v_{T',S'}^G u_{T,S}^G = 0$  by Theorem 4.4. Moreover:

$$\begin{aligned} \sum_{(T,S) \in [\mathcal{M}(G)]} \epsilon_{T,S}^G &= \sum_{(T,S) \in [\mathcal{M}(G)]} \frac{1}{|N_G(T,S):T|} u_{T,S}^G v_{T,S}^G \\ &= \sum_{(T,S) \in \mathcal{M}(G)} \frac{1}{|G:T|} u_{T,S}^G v_{T,S}^G \\ &= \sum_{(T,S) \in \mathcal{M}(G)} \frac{1}{|G:T|} \text{Indinf}_{T/S}^G \varphi_1^{T/S} \text{Defres}_{T/S}^G \end{aligned}$$

Now  $\varphi_1^{T/S} = \widetilde{e_{T/S}^{T/S}} f_{T/S}$  by Proposition 3.6, where

$$f_{T/S} = \sum_{\substack{N/S \trianglelefteq (T/S) \\ N/S \leq \Phi(T/S)}} \mu_{\trianglelefteq G}(\mathbf{1}, N/S) [((T/S) \times (T/S)) / \Delta_{N/S}(T/S)] \quad .$$

Hence  $\varphi_1^{T/S} = \widetilde{e_{T/S}^{T/S}} \text{Def}_{T/S}^T \text{Inf}_{T/S}^T f_{T/S}$ , and

$$\sum_{(T,S) \in [\mathcal{M}(G)]} \epsilon_{T,S}^G = \sum_{(T,S) \in \mathcal{M}(G)} \frac{1}{|G:T|} \text{Ind}_T^G \text{Inf}_{T/S}^T \widetilde{e_{T/S}^{T/S}} \text{Def}_{T/S}^T \text{Inf}_{T/S}^T f_{T/S} \text{Def}_{T/S}^T \text{Res}_T^G \quad .$$

Now  $\text{Inf}_{T/S}^T \widetilde{e_{T/S}^{T/S}} \text{Def}_{T/S}^T = \widetilde{\text{Inf}_{T/S}^T e_{T/S}^{T/S}}$ , and  $\text{Inf}_{T/S}^T e_{T/S}^{T/S}$  is equal to the sum over subgroups  $X$  or  $T$  such that  $XS = T$ , up to conjugation, of the idempotents  $e_X^T$ . Since  $S \leq \Phi(T)$ , the only subgroup  $X$  of  $T$  such that  $XS = T$  is  $T$  itself. Hence

$$\text{Inf}_{T/S}^T \widetilde{e_{T/S}^{T/S}} \text{Def}_{T/S}^T = \widetilde{e_T^T} \quad .$$

On the other hand

$$\text{Inf}_{T/S}^T[((T/S) \times (T/S))/\Delta_{N/S}(T/S)]\text{Def}_{T/S}^T = [(T \times T)/\Delta_N(T)] .$$

It follows that the sum  $\Sigma = \sum_{(T,S) \in [\mathcal{M}(G)]} \epsilon_{T,S}^G$  is equal to

$$\begin{aligned} \Sigma &= \sum_{(T,S) \in \mathcal{M}(G)} \frac{1}{|G:T|} \text{Ind}_T^G \widetilde{e}_T^T \sum_{\substack{N \trianglelefteq T \\ S \leq N \leq \Phi(T)}} \mu_{\trianglelefteq T}(S, N) [(T \times T)/\Delta_N(T)] \text{Res}_T^G \\ &= \sum_{(T,S) \in \mathcal{M}(G)} \frac{1}{|G:T|} \text{Ind}_T^G \widetilde{e}_T^T \varphi_S^T \text{Res}_T^G \quad [\text{by definition of } \varphi_S^T] \\ &= \sum_{T \leq G} \frac{1}{|G:T|} \text{Ind}_T^G \widetilde{e}_T^T \sum_{\substack{S \trianglelefteq T \\ S \leq \Phi(T)}} \varphi_S^T \text{Res}_T^G \\ &= \sum_{T \leq G} \frac{1}{|G:T|} \text{Ind}_T^G \widetilde{e}_T^T \text{Res}_T^G \quad [\text{by Theorem 3.8}] \\ &= \sum_{T \leq G} \frac{1}{|G:T|} \widetilde{\text{Ind}_T^G \widetilde{e}_T^T} \quad [\text{by Lemma 2.10}] \\ &= \sum_{T \leq G} \frac{1}{|G:N_G(T)|} \widetilde{e}_T^G \quad [\text{by (2.12)}] \\ &= \sum_{T \in [s_G]} \widetilde{e}_T^G = \widetilde{G/G} = [(G \times G)/\Delta(G)] . \end{aligned}$$

So the sum  $\Sigma$  is equal to the identity of  $RB(G, G)$ . Since  $\epsilon_{T,S}^G \epsilon_{T',S'}^G = 0$  if  $(T, S)$  and  $(T', S')$  are distinct elements of  $[\mathcal{M}(G)]$ , it follows that for any  $(T, S) \in [\mathcal{M}(G)]$

$$\epsilon_{T,S}^G = \epsilon_{T,S}^G \Sigma = (\epsilon_{T,S}^G)^2 ,$$

which completes the proof of the theorem.  $\square$

## 5. Application to biset functors

**5.1. Notation:** Let  $F$  be a biset functor over  $R$ . When  $G$  is a finite group with order invertible in  $R$ , we set

$$\delta_\Phi F(G) = \varphi_1^G F(G)$$

**5.2. Proposition:** *Let  $F$  be a biset functor over  $R$ . Then for any finite group  $G$  with order invertible in  $R$ , the  $R$ -submodule  $\delta_\Phi F(G)$  of  $F(G)$  is the set of elements  $u \in F(G)$  such that*

$$\begin{cases} \text{Res}_H^G u = 0 & \forall H < G \\ \text{Def}_{G/N}^G u = 0 & \forall N \trianglelefteq G, N \cap \Phi(G) \neq \mathbf{1} \end{cases}.$$

**Proof :** If  $u \in \delta_\Phi F(G) = \varphi_1^G F(G)$ , then  $\text{Res}_H^G u = 0$  for any proper subgroup  $H$  of  $G$ , and  $\text{Def}_{G/N}^G u = 0$  for any  $N \trianglelefteq G$  such that  $N \cap \Phi(G) \neq \mathbf{1}$ , by Corollary 3.7.

Conversely, if  $u \in F(G)$  fulfills the two conditions of the proposition, then  $\widetilde{e}_G^G u = u$ , because  $\widetilde{e}_G^G$  is equal to the identity element  $[(G \times G)/\Delta(G)]$  of  $RB(G, G)$ , plus a linear combination of elements of the form  $[(G \times G)/\Delta(H)] = \text{Ind}_H^G \circ \text{Res}_H^G$ , for proper subgroups  $H$  of  $G$ . Similarly  $\text{Inf}_{G/N}^G \text{Def}_{G/N}^G u = 0$  for any non-trivial normal subgroup of  $G$  contained in  $\Phi(G)$ , thus  $\varphi_1^G u = u$ .  $\square$

**5.3. Remark:** Since  $\text{Def}_{G/N}^G = \text{Def}_{G/N}^{G/M} \circ \text{Def}_{G/M}^G$ , where  $M = N \cap \Phi(G)$ , saying that  $\text{Def}_{G/N}^G u = 0$  for any  $N \trianglelefteq G$  with  $N \cap \Phi(G) \neq \mathbf{1}$  is equivalent to saying that  $\text{Def}_{G/N}^G u = 0$  for any non trivial normal subgroup  $N$  of  $G$  contained in  $\Phi(G)$ .

**5.4. Theorem:** *Let  $F$  be a biset functor over  $R$ . Then for any finite group  $G$  with order invertible in  $R$ , the maps*

$$\begin{aligned} F(G) &\xrightleftharpoons{\quad} \bigoplus_{(T,S) \in [\mathcal{M}(G)]} (\delta_\Phi F(T/S))^{N_G(T,S)/T} \\ w &\xrightarrow{V} \bigoplus_{(T,S)} \frac{1}{|N_G(T,S):T|} v_{T,S}^G w \\ \sum_{(T,S)} u_{T,S}^G w_{T,S} &\xleftarrow{U} \bigoplus_{(T,S)} w_{T,S} \end{aligned}$$

*are well defined isomorphisms of  $R$ -modules, inverse to one other.*

**Proof :** We have first to check that if  $w \in F(G)$ , then the element  $v_{T,S}^G w$  of  $\varphi_1^{T/S} F(T/S) = \delta_\Phi F(T/S)$  is invariant under the action of  $N_G(T, S)/T$ . But for any  $g \in N_G(T/S)$

$$\text{Iso}(c_g) v_{T,S}^G = v_{gT, gS}^G \text{Iso}(c_g) = v_{T,S}^G \text{Iso}(c_g),$$

where  $\text{Iso}(c_g) : F(G) \rightarrow F(G)$  on the right hand side is conjugation by  $g$ , that is an inner automorphism, hence the identity map, for  $g \in G$ .

Now for  $w \in F(G)$

$$\begin{aligned} UV(w) &= \sum_{(T,S) \in [\mathcal{M}(G)]} \frac{1}{|N_G(T,S):T|} u_{T,S}^G v_{T,S}^G w \\ &= \sum_{(T,S) \in [\mathcal{M}(G)]} \epsilon_{T,S}^G w = w, \end{aligned}$$

so  $UV$  is the identity map of  $F(G)$ .

Conversely, if  $w_{T,S} \in (\delta_\Phi F(T/S))^{N_G(T,S)/T}$ , for  $(T, S) \in [\mathcal{M}(G)]$ , then

$$\begin{aligned} VU\left(\bigoplus_{(T,S) \in [\mathcal{M}(G)]} w_{T,S}\right) &= \bigoplus_{(T,S) \in [\mathcal{M}(G)]} \sum_{(T',S') \in [\mathcal{M}(G)]} \frac{1}{|N_G(T,S):T|} v_{T,S}^G u_{T',S'}^G w_{T',S'} \\ &= \bigoplus_{(T,S) \in [\mathcal{M}(G)]} \frac{1}{|N_G(T,S):T|} v_{T,S}^G u_{T,S}^G w_{T,S} \\ &= \bigoplus_{(T,S) \in [\mathcal{M}(G)]} \frac{1}{|N_G(T,S):T|} \sum_{g \in N_G(T,S)/T} \text{Iso}(c_g) w_{T,S} \\ &= \bigoplus_{(T,S) \in [\mathcal{M}(G)]} w_{T,S}, \end{aligned}$$

so  $VU$  is also equal to the identity map.  $\square$

## 6. Atoric $p$ -groups

For the remainder of the paper, we denote by  $p$  a (fixed) prime number.

### 6.1. Notation and Definition:

- If  $P$  is a finite  $p$ -group, let  $\Omega_1 P$  denote the subgroup of  $P$  generated by the elements of order  $p$ .
- A finite  $p$ -group  $P$  is called *atoric* if it does not admit any decomposition  $P = E \times Q$ , where  $E$  is a non-trivial elementary abelian  $p$ -group. Let  $\mathcal{At}_p$  denote the class of atoric  $p$ -groups, and let  $[\mathcal{At}_p]$  denote a set of representatives of isomorphism classes in  $\mathcal{At}_p$ .

The terminology “atoric” is inspired by [14], where elementary abelian  $p$ -groups are called *p-tori*. Atoric  $p$ -groups have been considered (without naming them) in [6], Example 5.8.

**6.2. Lemma:** *Let  $P$  be a finite  $p$ -group, and  $N$  be a normal subgroup of  $P$ . The following conditions are equivalent:*

1.  $N \cap \Phi(P) = \mathbf{1}$
2.  $N$  is elementary abelian and central in  $P$ , and admits a complement in  $P$ .
3.  $N$  is elementary abelian and there exists a subgroup  $Q$  of  $P$  such that  $P = N \times Q$ .

**Proof :**

$\boxed{1 \Rightarrow 3}$  Let  $N \trianglelefteq P$  with  $N \cap \Phi(P) = \mathbf{1}$ . Then  $N$  maps injectively in the elementary abelian  $p$ -group  $P/\Phi(P)$ , so  $N$  is elementary abelian. Let  $Q/\Phi(P)$  be a complement of  $N\Phi(P)/\Phi(P)$  in  $P/\Phi(P)$ . Then  $Q \geq \Phi(P) \geq [P, P]$ , so  $Q$  is normal in  $P$ . Moreover  $Q \cdot N = P$  and  $Q \cap N\Phi(P) = (Q \cap N)\Phi(P) = \Phi(P)$ , thus  $Q \cap N \leq \Phi(P) \cap N = \mathbf{1}$ . Now  $N$  and  $Q$  are normal subgroups of  $P$  which intersect trivially, hence they centralize each other. It follows that  $P = N \times Q$ .

$\boxed{3 \Rightarrow 2}$  This is clear.

$\boxed{2 \Rightarrow 1}$  If  $P = N \cdot Q$  for some subgroup  $Q$  of  $P$ , and if  $N$  is central in  $P$ , then  $P = N \times Q$ . Thus  $\Phi(P) = \mathbf{1} \times \Phi(Q)$ , as  $N$  is elementary abelian. Then  $N \cap \Phi(P) \leq N \cap Q = \mathbf{1}$ .  $\square$

**6.3. Lemma:** *Let  $P$  be a finite  $p$ -group. The following conditions are equivalent:*

1.  $P$  is atoric.
2. If  $N \trianglelefteq P$  and  $N \cap \Phi(P) = \mathbf{1}$ , then  $N = \mathbf{1}$ .
3.  $\Omega_1 Z(P) \leq \Phi(P)$ .

**Proof :**

$\boxed{1 \Rightarrow 2}$  Suppose that  $P$  is atoric. Let  $N \trianglelefteq P$  with  $N \cap \Phi(P) = \mathbf{1}$ . Then by Lemma 6.2, the group  $N$  is elementary abelian and there exists a subgroup  $Q$  of  $P$  such that  $P = N \times Q$ . Hence  $N = \mathbf{1}$ .

$\boxed{2 \Rightarrow 3}$  Suppose now that Assertion 2 holds. If  $x$  is a central element of order  $p$  of  $P$ , then the subgroup  $N$  of  $P$  generated by  $x$  is normal in  $P$ , and non trivial. Then  $N \cap \Phi(P) \neq \mathbf{1}$ , hence  $N \leq \Phi(P)$  since  $N$  has order  $p$ , thus  $x \in \Phi(P)$ .

$\boxed{3 \Rightarrow 1}$  Finally, if Assertion 3 holds, and if  $P = E \times Q$  for some subgroups  $E$  and  $Q$  of  $P$  with  $E$  elementary abelian, then  $\Phi(P) = \mathbf{1} \times \Phi(Q)$ . Moreover  $E \leq \Omega_1 Z(P) \leq \Phi(P) \leq Q$ , so  $E = E \cap Q = \mathbf{1}$ , and  $P$  is atoric.  $\square$



**6.4. Proposition:** *Let  $P$  be a finite  $p$ -group, and  $N$  be a maximal normal subgroup of  $P$  such that  $N \cap \Phi(P) = \mathbf{1}$ . Then:*

1. *The group  $N$  is elementary abelian and there exists a subgroup  $T$  of  $P$  such that  $P = N \times T$ .*
2. *The group  $P/N \cong T$  is atoric.*
3. *If  $Q$  is an atoric  $p$ -group and  $s : P \twoheadrightarrow Q$  is a surjective group homomorphism, then  $s(T) = Q$ . In particular  $Q$  is isomorphic to a quotient of  $T$ .*

**Proof :** (1) This follows from Lemma 6.2.

(2) By (1), there exists  $T \leq P$  such that  $P = N \times T$ . In particular  $P/N \cong T$ . Now if  $T = E \times S$ , for some subgroups  $E$  and  $S$  of  $T$  with  $E$  elementary abelian, then  $P \cong P_1 = (N \times E) \times S$ , and  $N \times E$  is an elementary abelian normal subgroup of  $P_1$  which intersects trivially  $\Phi(P_1) = \Phi(S)$ . By maximality of  $N$ , it follows that  $E = \mathbf{1}$ , so  $T \cong P/N$  is atoric.

(3) Let  $s : P \twoheadrightarrow Q$  be a surjective group homomorphism, where  $Q$  is atoric. By (1), the group  $N$  is elementary abelian, and there exists a subgroup  $T$  of  $P$  such that  $P = N \times T$ . Then  $T \cong P^\circledast$ , and  $\Phi(P) = \Phi(T)$ . Moreover  $s(\Phi(P)) = \Phi(Q)$  as  $P$  is a  $p$ -group, and  $s(Z(P)) \leq Z(Q)$  as  $s$  is surjective. It follows that  $s(N)$  is an elementary abelian central subgroup of  $Q$ , so  $s(N) \leq \Phi(Q)$  since  $Q$  is atoric, by Lemma 6.3. Now  $s(P) = Q = s(N)s(T)$ , thus  $Q = \Phi(Q)s(T)$ , and  $s(T) = Q$ , as was to be shown.  $\square$

**6.5. Notation:** *When  $P$  is a finite  $p$ -group, and  $N$  is a maximal normal subgroup of  $P$  such that  $N \cap \Phi(P) = \mathbf{1}$ , we set  $P^\circledast = P/N$ .*

By Proposition 6.4, the group  $P^\circledast$  does not depend on the choice of  $N$ , up to isomorphism: it is the greatest atoric quotient of  $P$ , in the sense that any atoric quotient of  $P$  is isomorphic to a quotient of  $P^\circledast$ . In particular  $P^\circledast$  is trivial if and only if  $P$  is elementary abelian.

**6.6. Proposition:** *Let  $s : P \twoheadrightarrow Q$  be a surjective group homomorphism. Then  $P^\circledast \cong Q^\circledast$  if and only if  $\text{Ker}(s) \cap \Phi(P) = \mathbf{1}$ .*

**Proof :** Let  $E$  be a maximal normal subgroup of  $P$  such that  $E \cap \Phi(P) = \mathbf{1}$ , and  $T$  be a subgroup of  $P$  such that  $P = E \times T$ . Then  $E$  is elementary abelian, and  $\Phi(P) = \Phi(T)$ . Let  $\pi : Q \rightarrow Q^\circledast$  be the canonical projection. By definition, we have  $T \cong P^\circledast$ , and by Proposition 6.4, we have  $\pi \circ s(T) = Q^\circledast$ .

Hence  $Q^\oplus$  is a quotient of  $P^\oplus$ , and  $P^\oplus \cong Q^\oplus$  if and only if the map  $\pi \circ s$  induces an isomorphism from  $T$  to  $Q^\oplus$ , that is if  $\text{Ker}(\pi \circ s) \cap T = \mathbf{1}$ . This implies  $\text{Ker}(s) \cap T = \mathbf{1}$ , hence  $\text{Ker}(s) \cap \Phi(P) = \mathbf{1}$ .

Conversely, if  $\text{Ker}(s) \cap \Phi(P) = \mathbf{1}$ , then  $\text{Ker}(s) \cap \Phi(T) = \mathbf{1}$ . Now the group  $M = \text{Ker}(s) \cap T$  is a normal subgroup of  $T$  such that  $M \cap \Phi(T) = \mathbf{1}$ . Since  $T$  is atoric, it follows from Lemma 6.3 that  $M = \mathbf{1}$ , hence  $s(T) \cong T$ . Now  $Q = s(E)s(T)$ , and  $s(E)$  is a central elementary abelian subgroup of  $Q$ , since  $s$  is surjective. Let  $F$  be a complement of  $G = s(E) \cap s(T)$  in  $s(E)$ . Then  $Q = (F \cdot G)s(T) = F \cdot s(T)$ , thus  $Q = F \times s(T)$  since  $F$  is central in  $Q$ . It follows that  $s(T)$  is a quotient of  $Q$ . Since  $s(T) \cong T \cong P^\oplus$  is atoric, the group  $P^\oplus$  is isomorphic to a quotient of  $Q^\oplus$ , thus  $P^\oplus \cong Q^\oplus$ .  $\square$

**6.7. Proposition:** *Let  $P$  be a finite  $p$ -group, and  $Q$  be a subquotient of  $P$ . Then  $Q^\oplus$  is a subquotient of  $P^\oplus$ .*

**Proof :** Let  $(V, U)$  be a section of  $P$  such that  $V/U \cong Q$ . Then  $Q^\oplus$  is isomorphic to a quotient of  $V^\oplus$ , by Lemma 6.4. Hence it suffices to prove that  $V^\oplus$  is a subquotient of  $P^\oplus$ .

Let  $E$  be a maximal normal subgroup of  $P$  such that  $E \cap \Phi(P) = \mathbf{1}$ , and  $T$  be a subgroup of  $P$  such that  $P = E \times T$ . Then  $V \leq E \times T$ , so there exist a subgroup  $F$  of  $E$ , a subgroup  $X$  of  $T$ , a group  $Y$ , and surjective group homomorphisms  $\alpha : F \rightarrow Y$  and  $\beta : X \rightarrow Y$  such that

$$V = \{(f, x) \in F \times X \mid \alpha(f) = \beta(x)\}.$$

Now  $F \leq E$  is elementary abelian. If  $(f, x), (f', x') \in V$ , then  $[(f, x), (f', x')] = (1, [x, x'])$ , so  $[V, V] \leq \mathbf{1} \times [X, X]$ . Conversely if  $x, x' \in X$ , then there exist  $f, f' \in F$  such that  $\alpha(f) = \beta(x)$  and  $\alpha(f') = \beta(x')$ , i.e.  $(f, x), (f', x') \in V$ . Then  $[(f, x), (f', x')] = (1, [x, x'])$ , and it follows that  $[V, V] = \mathbf{1} \times [X, X]$ . Similarly, if  $(f, x) \in V$ , then  $(f, x)^p = (1, x^p)$ . Conversely, if  $x \in X$ , then there exists  $f \in F$  such that  $\alpha(f) = \beta(x)$ , i.e.  $(f, x) \in V$ , and  $(1, x^p) = (f, x)^p$ . It follows that  $\Phi(V) = \mathbf{1} \times \Phi(X)$ .

Now  $N = \text{Ker}(\alpha) \times \mathbf{1}$  is a normal subgroup of  $V$ , and  $N \cap \Phi(V) = \mathbf{1}$ . By Proposition 6.6, it follows that  $V^\oplus \cong (V/N)^\oplus$ . Moreover the group homomorphism  $(f, x) \in V \mapsto x \in X$  is surjective with kernel  $N$ , hence  $V/N \cong X$ . It follows that  $V^\oplus \cong X^\oplus$  is isomorphic to a quotient of the subgroup  $X$  of  $T \cong P^\oplus$ . Hence  $V^\oplus$  is a subquotient of  $P^\oplus$ , as was to be shown.  $\square$

**6.8. Proposition:** *Let  $P$  be a finite  $p$ -group, let  $N$  be a normal subgroup of  $P$  such that  $P/N \cong P^\oplus$ , and let  $Q$  be a subgroup of  $P$ . The following are*

equivalent:

1.  $Q^\circ \cong P^\circ$ .
2.  $QN = P$ .
3. There exists a central elementary abelian subgroup  $E$  of  $P$  such that  $P = EQ$ .
4. There exists an elementary abelian subgroup  $E$  of  $P$  such that  $P = E \times Q$ .

**Proof :**  $[1 \Rightarrow 2]$  Suppose  $Q^\circ \cong P^\circ$ . We have  $N \cap \Phi(T) = \mathbf{1}$ , by Proposition 6.6. Moreover  $\Phi(Q) \leq \Phi(P)$ , as  $P$  is a  $p$ -group. Setting  $M = N \cap Q$ , we have  $M \cap \Phi(Q) = \mathbf{1}$ , so  $(Q/M)^\circ \cong Q^\circ \cong P^\circ$ . But  $\bar{Q} = Q/M \cong QN/N$  is a subgroup of  $P/N \cong P^\circ$ , and moreover there exists an elementary abelian subgroup  $E$  of  $\bar{Q}$  such that  $\bar{Q} \cong E \times \bar{Q}^\circ \cong E \times P^\circ$ . Hence  $E = \mathbf{1}$  and  $\bar{Q} \cong QN/N \cong P/N$ , so  $QN = P$ , as was to be shown.

$[2 \Rightarrow 3]$  We have  $N \cap \Phi(P) = \mathbf{1}$ , by Proposition 6.6. Hence  $N$  is elementary abelian, and central in  $P$ , and 2 implies 3.

$[2 \Rightarrow 3]$  Let  $E$  be an elementary abelian central subgroup of  $P$  such that  $P = EQ$ . Let  $F$  be a complement of  $E \cap Q$  in  $E$ . Then  $F$  is elementary abelian and central in  $P$ . Moreover  $QF = QE = P$ , and  $Q \cap F = \mathbf{1}$ . Hence  $P = F \times Q$ .

$[4 \Rightarrow 1]$  If  $P = E \times Q$  and  $E$  is elementary abelian, then  $\Phi(P) = \mathbf{1} \times \Phi(Q)$ . Thus  $E \cap \Phi(P) = \mathbf{1}$ , so  $(P/E)^\circ \cong P^\circ$  by Proposition 6.6, and  $Q^\circ \cong P^\circ$ .  $\square$

## 6.9. Proposition:

1. Let  $L$  be an atoric  $p$ -group, let  $P = E \times L$  and  $Q = F \times L$ , where  $E$  and  $F$  are elementary abelian  $p$ -groups, and let  $s : P \rightarrow Q$  be a group homomorphism. Then  $s$  is surjective if and only if there exist a surjective group homomorphism  $a : E \rightarrow F$ , group homomorphisms  $b : L \rightarrow F$  and  $c : E \rightarrow \Omega_1 Z(L)$ , and an automorphism  $d$  of  $L$  such that

$$\forall (e, l) \in E \times L, \quad s(e, l) = (a(e)b(l), c(e)d(l)) .$$

Moreover in this case  $b \circ c(e) = 1$  for any  $e \in E$ , and  $s$  is an isomorphism if and only if  $a$  is an isomorphism.

2. Let  $P$  be a finite  $p$ -group. For a group homomorphism

$$\lambda : P \rightarrow \Omega_1 Z(P) \cap \Phi(P) ,$$

let  $\alpha_\lambda : P \rightarrow P$  be defined by  $\alpha(x) = x\lambda(x)$ , for  $x \in P$ . Then  $\alpha_\lambda$  is an automorphism of  $P$ .

3. Let  $P$  be a finite  $p$ -group, and let  $P = E \times Q$ , where  $Q$  is atoric and  $E$  is elementary abelian. Then the correspondence  $\lambda \mapsto \alpha_\lambda(E)$  is a bijection from the set of group homomorphisms  $\lambda : P \rightarrow \Omega_1 Z(P) \cap \Phi(P)$  such that  $Q \leq \text{Ker } \lambda$  to the set of subgroups  $N$  of  $P$  such that  $P = N \times Q$ .

**Proof :** (1) If  $s$  is surjective, then  $s(E)$  is central in  $Q$ , so  $s(E) \leq \Omega_1 Z(Q) = F \times \Omega_1 Z(L)$ . Hence there exists group homomorphisms  $a : E \rightarrow F$  and  $c : E \rightarrow \Omega_1 Z(L)$  such that  $s(e, 1) = (a(e), c(e))$ , for any  $e \in E$ . Let  $b : L \rightarrow F$  and  $d : L \rightarrow L$  be the group homomorphisms defined by  $s(1, l) = (b(l), d(l))$ , for  $l \in L$ . Then  $s(e, l) = s(e, 1)s(1, l) = (a(e)b(l), c(e)d(l))$  for all  $(e, l) \in P$ . Moreover  $b \circ c(e) = 1$  for any  $e \in E$ , since  $c(E) \leq \Omega_1 Z(L) \leq \Phi(L)$ , as  $L$  is atoric, and  $\Phi(L) \leq \text{Ker } b$ , as  $F$  is elementary abelian.

Now the composition of  $s$  with the projection  $F \times L \rightarrow L$  is surjective, hence  $s(1 \times L) = L$  by Proposition 6.4. In other words  $d$  is surjective, hence it is an automorphism of  $L$ .

Since  $s$  is surjective, for any  $(f, y) \in Q$ , there exists  $(e, x) \in P$  such that  $a(e)b(x) = f$  and  $c(e)d(x) = y$ . The latter gives  $x = d^{-1}(c(e)^{-1}y)$ . Then  $b(x) = bd^{-1}(c(e)^{-1})bd^{-1}(y)$ , and  $bd^{-1}(c(e)^{-1}) = 1$  since  $d^{-1}(c(e)^{-1}) \in d^{-1}\Omega_1 Z(L) = \Omega_1 Z(L)$ , and  $\Omega_1 Z(L) \leq \Phi(L) \leq \text{Ker } b$ . Then  $b(x) = bd^{-1}(y)$ , and  $f = a(e)bd^{-1}(y)$ . In particular, taking  $y = 1$ , we get that for any  $f \in L$ , there exists  $e \in E$  such that  $f = a(e)$ . In other words  $a$  is surjective.

Conversely, given a surjective group homomorphism  $a : E \rightarrow F$ , a group homomorphism  $b : L \rightarrow F$ , a group homomorphism  $c : E \rightarrow \Omega_1 Z(L)$ , and an automorphism  $d$  of  $L$ , we can define  $s : P \rightarrow Q$  by  $s(e, x) = (a(e)b(x), c(e)d(x))$ , for  $(e, x) \in P$ . This is clearly a group homomorphism, as  $F$  is abelian, and the image of  $c$  is central in  $L$ . We have again  $\Omega_1 Z(L) \leq \Phi(L) \leq \text{Ker } b$ , since  $F$  is elementary abelian. If  $(f, y) \in Q$ , we can choose an element  $e \in E$  such that  $f = a(e)bd^{-1}(y)$ , and then set  $x = d^{-1}(c(e)^{-1}y)$ , i.e.  $c(e)d(x) = y$ . We also have  $b(x) = bd^{-1}(y)$ , since  $d^{-1}(c(e)) \in \Omega_1 Z(L)$ , so  $f = a(e)b(x)$ . Hence  $s(e, x) = (f, y)$ , and  $s$  is surjective.

Finally if  $s$  is an isomorphism, then  $E \cong F$ , and then the surjection  $a$  is an isomorphism. Conversely, if  $a$  is an isomorphism, then  $E \cong F$ , so  $P \cong Q$ , and the surjection  $s$  is an isomorphism.

(2) Clearly  $\alpha_\lambda$  is a group homomorphism, since  $\lambda(P) \leq Z(P)$ . Moreover if  $x \in \text{Ker } \alpha_\lambda$ , then  $\lambda(x) = x$ , so  $x \in \Omega_1 Z(P) \cap \Phi(P) \leq \Phi(P) \leq \text{Ker } \lambda$ , since  $\Omega_1 Z(P) \cap \Phi(P)$  is elementary abelian. Thus  $x = 1$ , and  $\alpha_\lambda$  is injective. Hence it is an automorphism.

(3) Since  $P = E \times Q$ , we have  $\Omega_1 Z(P) = E \times \Omega_1 Z(Q)$ , and  $\Phi(P) = 1 \times \Phi(Q)$ .

So if  $\lambda$  is a group homomorphism from  $P$  to  $\Omega_1 Z(P) \cap \Phi(P)$  with  $Q \leq \text{Ker } \lambda$ , we have  $\lambda(e, l) = (1, \beta(e))$  for some group homomorphism  $\beta : E \rightarrow \Omega_1 Z(Q)$ . Then the group  $N = \alpha_\lambda(E) = \{(e, \beta(e)) \mid e \in E\}$  is central in  $P$ . Moreover  $N \cap Q = \mathbf{1}$ , and  $NQ = P$ , so  $P = N \times Q$ . Note that  $N$  determines the homomorphism  $\beta$ , hence also the homomorphism  $\lambda$ , so the map  $\lambda \mapsto \alpha_\lambda(E)$  is injective.

It is moreover surjective: indeed, if  $N$  is a subgroup of  $P = E \times Q$  such that  $P = N \times Q$ , then  $N \cong P/Q \cong E$  is elementary abelian, hence central in  $P$ . Since  $NQ = P$ , for any  $e \in E$ , there exists  $(a, b) \in N$  and  $q \in Q$  such that  $(e, 1) = (a, b)(1, q)$ , that is  $e = a$  and  $q = b^{-1}$ . In other words  $p_1(N) = E$ . Moreover  $N \cap Q = \mathbf{1}$ , so  $k_2(N) = \mathbf{1}$ . So for  $e \in E$ , there exists a unique  $x \in Q$  such that  $(e, x) \in N$ . Setting  $x = \beta(e)$ , we get a group homomorphism  $\beta : E \rightarrow Q$ , such that  $N = \{(e, \beta(e)) \mid e \in E\}$ . Since  $N$  is central in  $P$ , the image of  $\beta$  is contained in  $\Omega_1 Z(Q) \leq \Phi(Q)$ . Moreover  $\Omega_1 Z(P) = E \times \Omega_1 Z(Q)$ , and  $\Phi(P) = \mathbf{1} \times \Phi(Q)$ , so  $(\mathbf{1} \times \beta(E)) \leq \Omega_1 Z(P) \cap \Phi(P)$ . Setting  $\lambda(e, l) = (1, \beta(e))$ , we get a group homomorphism from  $P$  to  $\Omega_1 Z(P) \cap \Phi(P)$ , such that  $Q \leq \text{Ker } \lambda$ , and  $N = \alpha_\lambda(E)$ .  $\square$

## 7. Splitting the biset category of $p$ -groups, when $p \in R^\times$

**7.1. Notation and Definition:** Let  $RC_p$  denote the full subcategory of the biset category  $RC$  consisting of finite  $p$ -groups. A  $p$ -biset functor over  $R$  is an  $R$ -linear functor from  $RC_p$  to the category of  $R$ -modules. Let  $\mathcal{F}_{p,R}$  denote the full subcategory of  $\mathcal{F}_R$  consisting of  $p$ -biset functors over  $R$ .

In the statements below, we indicate by  $[p \in R^\times]$  the assumption that  $p$  is invertible in  $R$ .

**7.2. Theorem:**  $[p \in R^\times]$  Let  $P$  and  $Q$  be finite  $p$ -groups, let  $(T, S)$  be a minimal section of  $P$ , and  $(V, U)$  be a minimal section of  $Q$ . Then

$$\epsilon_{V,U}^Q RB(Q, P) \epsilon_{T,S}^P \neq \{0\} \implies (V/U)^\oplus \cong (T/S)^\oplus .$$

**Proof :** If  $\epsilon_{V,U}^Q RB(Q, P) \epsilon_{T,S}^P$ , there exists  $a \in RB(Q, P)$  such that

$$\epsilon_{V,U}^Q a \epsilon_{T,S}^P = \text{Indinf}_{V/U}^Q \varphi_1^{V/U} \text{Defres}_{V/U}^Q a \text{Indinf}_{T/S}^P \varphi_1^{T/S} \text{Defres}_{T/S}^P \neq 0 ,$$

and in particular the element  $b = \text{Defres}_{V/U}^Q a \text{Indinf}_{T/S}^P$  of  $RB(V/U, T/S)$  is such that  $\varphi_1^{V/U} b \varphi_1^{T/S} \neq 0$ . It follows that there is a subgroup  $L$  of the

product  $(V/U) \times (T/S)$  such that

$$\varphi_1^{V/U} [((V/U) \times (T/S))/L] \varphi_1^{T/S} \neq 0 .$$

Then Theorem 3.8 implies that  $p_1(L) = V/U$ ,  $k_1(L) \cap \Phi(V/U) = \mathbf{1}$ ,  $p_2(L) = T/S$ , and  $k_2(L) \cap \Phi(T/S) = \mathbf{1}$ . By Proposition 6.6, it follows that

$$(V/U)^\oplus \cong (p_1(L)/k_1(L))^\oplus \cong (p_2(L)/k_2(L))^\oplus \cong (T/S)^\oplus ,$$

as was to be shown.  $\square$

**7.3. Notation:**  $[p \in R^\times]$  Let  $L$  be an atoric  $p$ -group. If  $P$  is a finite  $p$ -group, we set

$$b_L^P = \sum_{\substack{(T,S) \in [\mathcal{M}(G)] \\ (T/S)^\oplus \cong L}} \epsilon_{T,S}^P .$$

**7.4. Theorem:**  $[p \in R^\times]$

1. Let  $L$  be an atoric  $p$ -group, and  $P$  be a finite  $p$ -group. Then  $b_L^P \neq 0$  if and only if  $L \subseteq P^\oplus$ .
2. Let  $L$  and  $M$  be atoric  $p$ -groups, and let  $P$  and  $Q$  be finite  $p$ -groups. If  $b_M^Q RB(Q, P) b_L^P \neq \{0\}$ , then  $M \cong L$ .
3. Let  $L$  be an atoric  $p$ -group, and let  $P$  and  $Q$  be finite  $p$ -groups. Then for any  $a \in RB(Q, P)$

$$b_L^Q a = a b_L^P .$$

4. The family of elements  $b_L^P \in RB(P, P)$ , for finite  $p$ -groups  $P$ , is an idempotent endomorphism  $b_L$  of the identity functor of the category  $RC_p$  (i.e. an idempotent of the center of  $RC_p$ ). The idempotents  $b_L$ , for  $L \in [\mathcal{A}t_p]$ , are orthogonal, and their sum is equal to the identity element of the center of  $RC_p$ .
5. For a given finite  $p$ -group  $P$ , the elements  $b_L^P$ , for  $L \in [\mathcal{A}t_p]$  such that  $L \subseteq P^\oplus$ , are non zero orthogonal central idempotents of  $RB(P, P)$ , and their sum is equal to the identity of  $RB(P, P)$ .

**Proof :** (1) The idempotent  $b_L^P$  is non zero if and only if there exists a minimal section  $(T, S)$  of  $P$  such that  $(T/S)^\oplus \cong L$ . Then  $L \subseteq P^\oplus$ , by Proposition 6.7. Conversely, if  $L \subseteq P^\oplus$ , then  $L \subseteq P$ , and there exists a minimal section  $(T, S)$  of  $P$  such that  $T/S \cong L$ . Then  $(T/S)^\oplus \cong L^\oplus \cong L$ , so  $\epsilon_{T,S}^P$  appears in the sum defining  $b_L^P$ , thus  $b_L^P \neq 0$ .

(2) If  $b_M^Q RB(Q, P) b_L^P \neq \{0\}$ , then there exist a minimal section  $(V, U)$  of  $Q$  with  $(V/U)^\oplus \cong M$  and a minimal section  $(T, S)$  of  $P$  with  $(T/S)^\oplus \cong L$  such that  $\epsilon_{V,U}^Q RB(Q, P) \epsilon_{T,S}^P \neq 0$ . Then  $(V/U)^\oplus \cong (T/S)^\oplus$  by Theorem 7.2, that is  $M \cong L$ .

(3) The identity element of  $RB(P, P)$  is equal to the sum of the idempotents  $\epsilon_{T,S}^P$ , for  $(T, S) \in [\mathcal{M}(P)]$ . Grouping those idempotents  $\epsilon_{T,S}^P$  for which  $(T/S)^\oplus$  is isomorphic to a given  $L \in [\mathcal{A}t_p]$  shows that the identity element of  $RB(P, P)$  is equal to the sum of the idempotents  $b_L^P$ , for  $L \in [\mathcal{A}t_p]$  (and there are finitely many non zero  $b_L^P$ , by (1)). It follows that

$$\begin{aligned} b_M^Q a &= b_M^Q a \sum_{L \in [\mathcal{A}t_p]} b_L^P = \sum_{L \in [\mathcal{A}t_p]} b_M^Q a b_L^P \\ &= b_M^Q a b_M^P \text{ [by (2)]} \\ &= \sum_{L \in [\mathcal{A}t_p]} b_L^Q a b_M^P \text{ [by (2)]} \\ &= a b_M^P, \end{aligned}$$

since  $\sum_{L \in [\mathcal{A}t_p]} b_L^Q$  is the identity element of  $RB(Q, Q)$ .

It follows that the family  $b_L^P$ , where  $P$  is a finite  $p$ -group, is an element  $b_L$  of the center of  $RC_p$ . Clearly  $b_L^2 = b_L$ , and if  $L$  and  $M$  are non isomorphic atoric  $p$ -groups, then  $b_L b_M = 0$ , by (2). Moreover the infinite sum  $\sum_{L \in [\mathcal{A}t_p]} b_L$  is actually locally finite, i.e. for each finite  $p$ -group  $P$ , the sum  $\sum_{L \in [\mathcal{A}t_p]} b_L^P$  has only finitely many non zero terms. The sum  $\sum_{L \in [\mathcal{A}t_p]} b_L$  is clearly equal to the identity endomorphism of the identity functor of  $RC_p$ .

(4) This is a straightforward consequence of (1) and (3).  $\square$

### 7.5. Corollary: $[p \in R^\times]$

1. Let  $L$  be an atoric  $p$ -group. For a  $p$ -biset functor  $F$ , the family of maps  $F(b_L^P) : F(P) \rightarrow F(P)$ , for finite  $p$ -groups  $P$ , is an endomorphism of  $F$ , denoted by  $F(b_L)$ .
2. If  $\theta : F \rightarrow G$  is a natural transformation of  $p$ -biset functors, the diagram

$$\begin{array}{ccc} F & \xrightarrow{F(b_L)} & F \\ \theta \downarrow & & \downarrow \theta \\ G & \xrightarrow{G(b_L)} & G \end{array}$$

is commutative. Hence the family of endomorphisms  $F(b_L)$ , for  $p$ -biset functors  $F$ , is an idempotent of the center of the category  $\mathcal{F}_{p,R}$ , denoted by  $\widehat{b}_L$ .

3. The idempotents  $\widehat{b}_L$ , for  $L \in [\mathcal{A}t_p]$ , are orthogonal idempotents of the center of  $\mathcal{F}_{p,R}$ , and their sum is the identity.
4. If  $F$  is a  $p$ -biset functor over  $R$ , let  $\widehat{b}_L F$  denote the image of the endomorphism  $F(b_L)$  of  $F$ . Then  $F = \bigoplus_{L \in [\mathcal{A}t_p]} \widehat{b}_L F$ .
5. Let  $\widehat{b}_L \mathcal{F}_{p,R}$  denote the full subcategory of  $\mathcal{F}_{p,R}$  consisting of functors  $F$  such that  $F = \widehat{b}_L F$ . Then  $\widehat{b}_L \mathcal{F}_{p,R}$  is an abelian subcategory of  $\mathcal{F}_{p,R}$ . Moreover the functor

$$F \in \mathcal{F}_{p,R} \mapsto (\widehat{b}_L F)_{L \in [\mathcal{A}t_p]} \in \prod_{L \in [\mathcal{A}t_p]} \widehat{b}_L \mathcal{F}_{p,R}$$

is an equivalence of categories.

**Proof :** All assertions are straightforward consequences of Theorem 7.4.  $\square$

**7.6. Notation:** For an atoric  $p$ -group  $L$ , let  $RC_p^L$  denote the full subcategory of  $RC_p$  consisting of the class  $\mathcal{Y}_L$  of finite  $p$ -groups  $P$  such that  $P^\circ \subseteq L$ . When  $p \in R^\times$ , Let moreover

$$b_L^+ = \sum_{\substack{H \in [\mathcal{A}t_p] \\ H \subseteq L}} b_H$$

be the sum of the idempotents  $b_H$  corresponding to atoric subquotients of  $L$ , up to isomorphism.

The class  $\mathcal{Y}_L$  is closed under taking subquotients, by Proposition 6.7. It follows that we can apply the results of Section 6 (Appendix) of [12]: if  $F$  is a  $p$ -biset functor over  $R$ , we can restrict  $F$  to an  $R$ -linear functor from  $RC_p^L$  to  $R\text{-Mod}$ . This yields a forgetful functor  $\mathcal{O}_{\mathcal{Y}_L} : \mathcal{F}_{p,R} \rightarrow \text{Fun}_R(RC_p^L, R\text{-Mod})$ . The right adjoint  $\mathcal{R}_{\mathcal{Y}_L}$  of this functor is described in full detail in Section 6 of [12], as follows: if  $G$  is an  $R$ -linear functor from  $RC_p^L$  to  $R\text{-Mod}$ , and  $P$  is a finite  $p$ -group, set

$$(7.7) \quad \mathcal{R}_{\mathcal{Y}_L}(G)(P) = \varprojlim_{(X,M) \in \Sigma_L(P)} G(X/M)$$

the inverse limit of modules  $G(X/M)$  on the set  $\Sigma_L(P)$  of sections  $(X, M)$



of  $P$  such that  $(X/M)^\circ \subseteq L$ , i.e. the set of sequences  $(l_{X,M})_{(X,M) \in \Sigma_L(P)}$  with the following properties:

1. if  $(X, M) \in \Sigma_L(P)$ , then  $l_{X,M} \in G(X/M)$ .
2. if  $(X, M), (Y, N) \in \Sigma_L(P)$  and  $M \leq N \leq Y \leq X$ , then

$$\text{Defres}_{Y/N}^{X/M} l_{X,M} = l_{Y,N} \quad .$$

3. if  $x \in P$  and  $(X, M) \in \Sigma_L(P)$ , then  ${}^x l_{X,M} = l_{xX, xM}$ .

Recall now that for finite groups  $P$  and  $Q$ , and for a finite  $(Q, P)$ -biset  $U$ , for a subgroup  $T$  of  $Q$  and an element  $u$  of  $U$ , the subgroup  $T^u$  of  $P$  is defined by  $T^u = \{x \in P \mid \exists t \in T \ tu = ux\}$ . By Lemma 6.4 of [12], if  $(T, S)$  is a section of  $Q$ , then  $(T^u, S^u)$  is a section of  $P$ , and  $T^u/S^u$  is a subquotient of  $T/S$ .

With this notation, when  $P$  and  $Q$  are finite  $p$ -groups, when  $U$  is a finite  $(Q, P)$ -biset, and  $l = (l_{X,M})_{(X,M) \in \Sigma_L(P)}$  is an element of  $\mathcal{R}_{\mathcal{Y}_L}(G)(P)$ , we denote by  $Ul$  the sequence indexed by  $\Sigma_L(Q)$  defined by

$$(Ul)_{Y,N} = \sum_{u \in [Y \backslash U/P]_L} (N \backslash Yu)(l_{Y^u, N^u})$$

where  $[Y \backslash U/P]$  is a set of representatives of  $(Y \times P)$ -orbits on  $U$ , and  $N \backslash Yu$  is viewed as a  $(Y/N, Y^u/N^u)$ -biset. It shown in Section 6 of [12] that  $Ul \in \mathcal{R}_{\mathcal{Y}_L}(G)(Q)$ , and that  $\mathcal{R}_{\mathcal{Y}_L}(G)$  becomes a  $p$ -biset functor in this way. Moreover<sup>1</sup>:

**7.8. Theorem:** [[12] Theorem 6.15] *The assignment  $G \mapsto \mathcal{R}_{\mathcal{Y}_L}(G)$  is an  $R$ -linear functor  $\mathcal{R}_{\mathcal{Y}_L}$  from  $\text{Fun}_R(R\mathcal{C}_p^L, R\text{-Mod})$  to  $\mathcal{F}_{p,R}$ , which is right adjoint to the forgetful functor  $\mathcal{O}_{\mathcal{Y}_L}$ . Moreover the composition  $\mathcal{O}_{\mathcal{Y}_L} \circ \mathcal{R}_{\mathcal{Y}_L}$  is isomorphic to the identity functor of  $\text{Fun}_R(R\mathcal{C}_p^L, R\text{-Mod})$ .*

**7.9. Theorem:** [ $p \in R^\times$ ] *For an atoric  $p$ -group  $L$ , let  $\widehat{b}_L^+ \mathcal{F}_{p,R}$  be the full subcategory of  $\mathcal{F}_{p,R}$  consisting of functors  $F$  such that  $\widehat{b}_L^+ F = F$ . Then the forgetful functor  $\mathcal{O}_{\mathcal{Y}_L}$  and its right adjoint  $\mathcal{R}_{\mathcal{Y}_L}$  restrict to quasi-inverse equivalences of categories*

$$\widehat{b}_L^+ \mathcal{F}_{p,R} \xrightleftharpoons[\mathcal{R}_{\mathcal{Y}_L}]{\mathcal{O}_{\mathcal{Y}_L}} \text{Fun}_R(R\mathcal{C}_p^L, R\text{-Mod}) \quad .$$

<sup>1</sup>In Theorem 6.15 of [12], only the case  $R = \mathbb{Z}$  is considered, but the proofs extend trivially to the case of an arbitrary commutative ring  $R$

**Proof :** First step: The first thing to check is that the image of the functor  $\mathcal{R}_{\mathcal{Y}_L}$  is contained in  $\widehat{b}_L^+ \mathcal{F}_{p,R}$ . We first prove that if  $H$  is an atoric  $p$ -group, if  $F \in \mathcal{F}_{p,R}$ , and if  $\mathcal{O}_{\mathcal{Y}_L}(\widehat{b}_H F) \neq 0$ , then  $H \sqsubseteq L$ : indeed in that case, there exists  $P \in \mathcal{Y}_L$  such that  $b_H^P F(P) \neq 0$ . In particular  $b_H^P \neq 0$  by Theorem 7.4, hence  $H \sqsubseteq P^\oplus$ . Since  $P^\oplus \sqsubseteq L$  as  $P \in \mathcal{Y}_L$ , it follows that  $H \sqsubseteq L$ , as claimed.

In particular

$$\mathcal{O}_{\mathcal{Y}_L}(F) = \mathcal{O}_{\mathcal{Y}_L} \left( \sum_{\substack{H \in [\mathcal{A}t_p] \\ H \sqsubseteq L}} \widehat{b}_H F \right) = \mathcal{O}_{\mathcal{Y}_L}(\widehat{b}_L^+ F) .$$

Set  $\mathcal{G}_p^L = \text{Fun}_R(R\mathcal{C}_p^L, R\text{-Mod})$ , and let  $G \in \mathcal{G}_p^L$ . Let  $H$  be an atoric  $p$ -group such that  $H \not\sqsubseteq L$ . If  $F \in \mathcal{F}_{p,R}$ , then

$$\begin{aligned} \text{Hom}_{\mathcal{F}_{p,R}}(F, \widehat{b}_H \mathcal{R}_{\mathcal{Y}_L}(G)) &= \text{Hom}_{\mathcal{F}_{p,R}}(\widehat{b}_H F, \widehat{b}_H \mathcal{R}_{\mathcal{Y}_L}(G)) \\ &= \text{Hom}_{\mathcal{F}_{p,R}}(\widehat{b}_H F, \mathcal{R}_{\mathcal{Y}_L}(G)) \\ &\cong \text{Hom}_{\mathcal{G}_p^L}(\mathcal{O}_{\mathcal{Y}_L}(\widehat{b}_H F), G) = \{0\} . \end{aligned}$$

So the functor  $F \mapsto \text{Hom}_{\mathcal{F}_{p,R}}(F, \widehat{b}_H \mathcal{R}_{\mathcal{Y}_L}(G))$  is the zero functor, and it follows from Yoneda's lemma that  $\widehat{b}_H \mathcal{R}_{\mathcal{Y}_L}(G) = 0$  if  $H \not\sqsubseteq L$ . In other words  $\mathcal{R}_{\mathcal{Y}_L}(G) = \widehat{b}_L^+ \mathcal{R}_{\mathcal{Y}_L}(G)$ , as was to be shown.

Second step: The first step shows that we have adjoint functors

$$\widehat{b}_L^+ \mathcal{F}_{p,R} \xrightleftharpoons[\mathcal{R}_{\mathcal{Y}_L}]{\mathcal{O}_{\mathcal{Y}_L}} \text{Fun}_R(R\mathcal{C}_p^L, R\text{-Mod}) = \mathcal{G}_p^L .$$

Moreover, the composition  $\mathcal{O}_{\mathcal{Y}_L} \circ \mathcal{R}_{\mathcal{Y}_L}$  is isomorphic to the identity functor, by Theorem 7.8. All we have to show is that the unit of the adjunction is also an isomorphism, in other words, that for any  $F \in \widehat{b}_L^+ \mathcal{F}_{p,R}$  and any finite  $p$ -group  $P$ , the natural map

$$(7.10) \quad \eta_P : F(P) \rightarrow \mathcal{R}_{\mathcal{Y}_L} \mathcal{O}_{\mathcal{Y}_L}(F)(P) = \varprojlim_{(X,M) \in \Sigma_L(P)} F(X/M)$$

sending  $u \in F(P)$  to the sequence  $(\text{Defres}_{X/M}^P u)_{(X,M) \in \Sigma_L(P)}$ , is an isomorphism.

The map  $\eta_P$  is injective: indeed, if  $u \in F(P)$ , then  $u = \sum_{\substack{H \in [\mathcal{A}t_p] \\ H \sqsubseteq L}} b_H^P u$ , as

$F = \widehat{b}_L^+ F$ . If  $\text{Defres}_{X/M}^P u = 0$  for any section  $(X, M)$  of  $P$  with  $(X/M)^\oplus \sqsubseteq L$ ,

then  $F(\epsilon_{T,S}^P)(u) = 0$  for any section  $(T, S)$  of  $P$  such that  $(T/S)^\oplus \subseteq L$ , by Proposition 4.6 and Proposition 6.7. In particular  $b_H^P u = 0$  for any atoric subquotient  $H$  of  $L$ , hence  $u = 0$ .

To prove that  $\eta_P$  is also surjective, we generalize the construction of Theorem A.2 of [11] (which is the case  $L = \mathbf{1}$ ), and we define, for an element  $v = (v_{X,M})_{(X,M) \in \Sigma_L(P)}$  in  $\mathcal{R}_{\mathcal{Y}_L} \mathcal{O}_{\mathcal{Y}_L}(F)(P)$ , an element  $u = \iota_P(v)$  of  $F(P)$  by

$$u = \frac{1}{|P|} \sum_{\substack{(T,S) \in \mathcal{M}(P) \\ (T/S)^\oplus \subseteq L}} \sum_{\substack{X \leq T, M \leq T \\ S \leq M \leq \Phi(T) \leq X \leq T}} |X| \mu(X, T) \mu_{\leq T}(S, M) \text{Indinf}_{X/M}^P v_{X,M} .$$

This yields an  $R$ -linear map  $\iota_P : \mathcal{R}_{\mathcal{Y}_L} \mathcal{O}_{\mathcal{Y}_L}(F)(P) \rightarrow F(P)$ .

For  $(Y, N) \in \Sigma_L(P)$ , set  $u_{Y,N} = \text{Defres}_{Y/N}^P u$ . Then:

$$u_{Y,N} = \sum_{\substack{(T,S) \in \mathcal{M}(P) \\ (T/S)^\oplus \subseteq L}} \sum_{\substack{X \leq T, M \leq T \\ S \leq M \leq \Phi(T) \leq X \leq T}} \frac{|X|}{|P|} \mu(X, T) \mu_{\leq T}(S, M) \text{Defres}_{Y/N}^P \text{Indinf}_{X/M}^P v_{X,M} .$$

Moreover

$$\text{Defres}_{Y/N}^P \text{Indinf}_{X/M}^P v_{X,M} = \sum_{g \in [Y \setminus P/X]} \text{Indinf}_{J_g/J'_g}^{Y/N} \text{Iso}(\phi_g) \text{Defres}_{I_g/I'_g}^{gX/gM} v_{X,N} ,$$

where  $J_g = N(Y \cap {}^g X)$ ,  $J'_g = N(Y \cap {}^g M)$ ,  $I_g = {}^g M(Y \cap {}^g X)$ ,  $I'_g = {}^g M(N \cap {}^g X)$ , and  $\phi_g$  is the isomorphism  $I_g/I'_g \rightarrow J_g/J'_g$  sending  $xI'_g$  to  $xJ'_g$ , for  $x \in Y \cap {}^g X$ . Hence

$$\begin{aligned} \text{Defres}_{Y/N}^P \text{Indinf}_{X/M}^P v_{X,M} &= \sum_{g \in [Y \setminus P/X]} \text{Indinf}_{J_g/J'_g}^{Y/N} \text{Iso}(\phi_g) v_{I_g, I'_g} \\ &= \frac{|Y \cap {}^g X|}{|Y||X|} \sum_{g \in P} \text{Indinf}_{J_g/J'_g}^{Y/N} \text{Iso}(\phi_g) v_{I_g, I'_g} . \end{aligned}$$

Thus

$$u_{Y,N} = \sum_{\substack{(T,S) \in \mathcal{M}(P) \\ (T/S)^\oplus \subseteq L \\ X \leq T, M \leq T \\ S \leq M \leq \Phi(T) \leq X \leq T \\ g \in P}} \frac{|Y \cap {}^g X|}{|P||Y|} \mu(X, T) \mu_{\leq T}(S, M) \text{Indinf}_{J_g/J'_g}^{Y/N} \text{Iso}(\phi_g) v_{I_g, I'_g} .$$

Now  $\mu(X, T) = \mu({}^g X, {}^g T)$  and  $\mu_{\leq T}(S, M) = \mu_{\leq {}^g T}({}^g S, {}^g M)$ , so summing over  $({}^g T, {}^g S, {}^g X, {}^g M)$  instead of  $(T, S, X, M)$  we get

$$u_{Y,N} = \sum_{\substack{(T,S) \in \mathcal{M}(P) \\ (T/S)^\oplus \subseteq L \\ X \leq T, M \leq T \\ S \leq M \leq \Phi(T) \leq X \leq T}} \frac{|Y \cap X|}{|Y|} \mu(X, T) \mu_{\leq T}(S, M) \text{Indinf}_{J_1/J'_1}^{Y/N} \text{Iso}(\phi_1) v_{I_1, I'_1} .$$

Setting  $W = Y \cap X$ , we have  $J_1 = NW$ ,  $J'_1 = N(W \cap M)$ ,  $I_1 = MW$ ,  $I'_1 = M(N \cap W)$ , and these four groups only depend on  $W$ , once  $M$  and  $N$  are given. Hence, for given  $T, S$  and  $M$ , we can group together the terms of the above summation for which  $Y \cap X$  is a given subgroup  $W$  of  $Y \cap T$ . This gives

$$u_{Y,N} = \sum_{\substack{(T,S) \in \mathcal{M}(P) \\ (T/S)^\oplus \subseteq L \\ M \trianglelefteq T \\ S \leq M \leq \Phi(T) \\ W \leq Y \cap T}} \left( \sum_{\substack{\Phi(T) \leq X \leq T \\ X \cap Y = W}} \mu(X, T) \right) \frac{|W|}{|Y|} \mu_{\leq T}(S, M) \text{Indinf}_{J_1/J'_1}^{Y/N} \text{Iso}(\phi_1) v_{I_1, I'_1}.$$

Moreover  $\sum_{\substack{\Phi(T) \leq X \leq T \\ X \cap Y = W}} \mu(X, T) = \sum_{\substack{X \leq T \\ X \cap (Y \cap T) = W}} \mu(X, T)$ , since  $\mu(X, T) = 0$  unless  $X \geq \Phi(T)$ , and the latter summation vanishes unless  $Y \cap T = T$ , by classical combinatorial lemmas ([15] Corollary 3.9.3). This gives:

$$u_{Y,N} = \sum_{\substack{(T,S) \in \mathcal{M}(P) \\ (T/S)^\oplus \subseteq L \\ M \trianglelefteq T \\ S \leq M \leq \Phi(T) \leq W \leq T \leq Y}} \frac{|W|}{|Y|} \mu(W, T) \mu_{\leq T}(S, M) \text{Indinf}_{J_1/J'_1}^{Y/N} \text{Iso}(\phi_1) v_{I_1, I'_1}.$$

Moreover in this summation  $J_1 = NW$ ,  $J'_1 = N(W \cap M) = NM$ ,  $I_1 = MW = W$ ,  $I'_1 = M(N \cap W) = MN \cap W$ . All these groups remain unchanged if we replace  $M$  by  $M(N \cap \Phi(T))$ , so for given  $T, S$  and  $W$ , we can group together those terms for which  $M(N \cap \Phi(T))$  is a given normal subgroup  $U$  of  $T$  with  $U \leq \Phi(T)$ . The sum  $\sum_{\substack{S \leq M \leq T \\ M(N \cap \Phi(T)) = U}} \mu_{\leq T}(S, M)$  is equal to 0 (by the

same above-mentioned classical combinatorial lemmas) unless  $N \cap \Phi(T) \leq S$ . Hence

$$u_{Y,N} = \sum_{\substack{(T,S) \in \mathcal{M}(P) \\ (T/S)^\oplus \subseteq L \\ U \trianglelefteq T \\ N \cap \Phi(T) \leq S \leq U \leq \Phi(T) \leq W \leq T \leq Y}} \frac{|W|}{|Y|} \mu(W, T) \mu_{\leq T}(S, U) \text{Indinf}_{J_1/J'_1}^{Y/N} \text{Iso}(\phi_1) v_{I_1, I'_1},$$

where  $J_1 = NW$ ,  $J'_1 = NU$ ,  $I_1 = W$ ,  $I'_1 = UN \cap W$ .

Now if  $N \cap \Phi(T) \leq S \leq \Phi(T) \leq T \leq Y$ , then  $(TN/N)^\oplus \subseteq (Y/N)^\oplus$ . Moreover the normal subgroup  $(N \cap T)/(N \cap \Phi(T))$  of  $T/(N \cap \Phi(T))$  intersects trivially the Frattini subgroup

$$\Phi\left(T/(N \cap \Phi(T))\right) = \Phi(T)(N \cap \Phi(T))/(N \cap \Phi(T)) \quad ,$$

so  $\left(T/(N \cap \Phi(T))\right)^{\oplus} \cong (T/(T \cap N))^{\oplus} \cong (TN/N)^{\oplus}$  by Proposition 6.6. Then  $(T/S)^{\oplus} \subseteq \left(T/(N \cap \Phi(T))\right)^{\oplus} \subseteq (TN/N)^{\oplus} \subseteq (Y/N)^{\oplus}$ . As  $(Y/N)^{\oplus} \subseteq L$  by assumption, it follows that

$$u_{Y,N} = \sum_{\substack{S \trianglelefteq T \leq Y \\ U \trianglelefteq T \\ N \cap \Phi(T) \leq S \leq U \leq \Phi(T) \leq W \leq T \leq Y}} \frac{|W|}{|Y|} \mu(W, T) \mu_{\trianglelefteq T}(S, U) \text{Indinf}_{J_1/J'_1}^{Y/N} \text{Iso}(\phi_1) v_{I_1, I'_1}.$$

Now the sum  $\sum_{\substack{S \trianglelefteq T \\ N \cap \Phi(T) \leq S \leq U}} \mu_{\trianglelefteq T}(S, U)$  is equal to zero unless  $U = N \cap \Phi(T)$ .

Hence

$$u_{Y,N} = \sum_{\Phi(T) \leq W \leq T \leq Y} \frac{|W|}{|Y|} \mu(W, T) \text{Indinf}_{J_1/J'_1}^{Y/N} \text{Iso}(\phi_1) v_{I_1, I'_1}.$$

For a given subgroup  $W$  of  $Y$ , the sum  $\sum_{\Phi(T) \leq W \leq T \leq Y} \mu(W, T)$  is equal to  $\sum_{W \leq T \leq Y} \mu(W, T)$  since  $\mu(W, T) = 0$  unless  $W \geq \Phi(T)$ , and the latter is equal to zero if  $W \neq Y$ , and to 1 if  $W = Y$ . Thus

$$u_{Y,N} = \frac{|Y|}{|Y|} \text{Indinf}_{J_1/J'_1}^{Y/N} \text{Iso}(\phi_1) v_{I_1, I'_1},$$

where  $J_1 = NY = Y$ ,  $J'_1 = N(Y \cap U) = N$ ,  $I_1 = Y$ ,  $I'_1 = UN \cap Y = N$ . Hence  $I_1 = J_1 = Y$  and  $I'_1 = J'_1 = N$ , so  $\phi_1$  is equal to the identity. It follows that  $u_{Y,N} = v_{Y,N}$  for any  $(Y, N) \in \Sigma_L(P)$ , so  $\eta_P(u) = v$ . This proves that the map  $\eta_P$  is surjective, hence an isomorphism, with inverse  $\iota_P$ . This completes the proof of Theorem 7.9.  $\square$

**7.11. Definition:** Let  $RC_p^{\#L}$  be the following category:

- The objects of  $RC_p^{\#L}$  are the finite  $p$ -groups  $P$  such that  $P^{\oplus} \cong L$ .
- If  $P$  and  $Q$  are finite  $p$ -groups such that  $P^{\oplus} \cong Q^{\oplus} \cong L$ , then

$$\text{Hom}_{RC_p^{\#L}}(P, Q) = RB(Q, P) / \sum_{L \not\leq S} RB(Q, S)B(S, P)$$

is the quotient of  $RB(Q, P)$  by the  $R$ -submodule generated by all morphisms from  $P$  to  $Q$  in  $RC_p$  which factor through a  $p$ -group  $S$  which do not admit  $L$  as a subquotient.

- The composition of morphisms in  $RC_p^{\#L}$  is induced by the composition of morphisms in  $RC_p$ .

**7.12. Remark:** Morphisms in  $RC_p$  which factor through a  $p$ -group  $S$  such that  $L \not\sqsubseteq S$  clearly generate a two-sided ideal, so the composition in  $RC_p^{\sharp L}$  is well defined. Moreover the category  $RC_p^{\sharp L}$  is  $R$ -linear. Let  $\text{Fun}_R(RC_p^{\sharp L}, R\text{-Mod})$  denote the category of  $R$ -linear functors from  $RC_p^{\sharp L}$  to the category  $R\text{-Mod}$  of  $R$ -modules.

**7.13. Lemma:** *Let  $p$  be a prime, and  $L$  be an atoric  $p$ -group. Let  $P$  and  $Q$  be finite  $p$ -groups.*

1. *If  $P^{\textcircled{a}} \cong L$  or  $Q^{\textcircled{a}} \cong L$ , and if  $M \leq (Q \times P)$ , then  $q(M)^{\textcircled{a}} \sqsubseteq L$ . Moreover  $q(M)^{\textcircled{a}} \cong L$  if and only if  $L \sqsubseteq q(M)$ .*
2. *If  $P^{\textcircled{a}} \cong Q^{\textcircled{a}} \cong L$ , then*

$$\text{Hom}_{RC_p^{\sharp L}}(P, Q) = RB(Q, P) / \sum_{S^{\textcircled{a}} \sqsubset L} RB(Q, S)B(S, P)$$

*is also the quotient of  $RB(Q, P)$  by the  $R$ -submodule generated by all morphisms from  $P$  to  $Q$  in  $RC_p$  which factor through a  $p$ -group  $S$  such that  $S^{\textcircled{a}}$  is a proper subquotient of  $L$ .*

3. *If  $P^{\textcircled{a}} \cong Q^{\textcircled{a}} \cong L$ , then  $\text{Hom}_{RC_p^{\sharp L}}(P, Q)$  has an  $R$ -basis consisting of the (images of the) transitive  $(Q, P)$ -bisets  $(Q \times P)/M$ , where  $M$  is a subgroup of  $(Q \times P)$  such that  $q(M)^{\textcircled{a}} \cong L$  (up to conjugation).*

**Proof:** (1) Indeed  $q(M)$  is a subquotient of  $P$ , and a subquotient of  $Q$ . Hence  $q(M)^{\textcircled{a}}$  is a subquotient of  $P^{\textcircled{a}}$  and a subquotient of  $Q^{\textcircled{a}}$ , thus  $q(M)^{\textcircled{a}} \sqsubseteq L^{\textcircled{a}} \cong L$ . Now suppose that  $q(M)^{\textcircled{a}} \cong L$ . Then  $L$  is a quotient of  $q(M)$ , so  $L \sqsubseteq q(M)$ . Conversely, if  $L \sqsubseteq q(M)$ , then  $L \cong L^{\textcircled{a}}$  is a subquotient of  $q(M)^{\textcircled{a}}$ , which is a subquotient of  $L$ . So  $q(M)^{\textcircled{a}} \cong L$ .

(2) Let  $S$  be a finite  $p$ -group such that  $L \not\sqsubseteq S$ , or equivalently  $L \not\sqsubseteq S^{\textcircled{a}}$ . Any element of  $RB(Q, S)B(S, P)$  is a linear combination of  $(Q, P)$ -bisets of the form  $(Q \times P)/(M * N)$ , for  $M \leq (Q \times S)$  and  $N \leq (S \times P)$ . This biset  $(Q \times P)/(M * N)$  also factors through  $T = q(M * N)$ , by 2.6. Moreover  $T$  is a subquotient of  $q(M)$  and  $q(N)$ , hence a subquotient of  $Q$ ,  $S$ , and  $P$ . Hence  $T^{\textcircled{a}} \sqsubseteq Q^{\textcircled{a}} \cong L$ , and  $T^{\textcircled{a}} \not\cong L$ , since  $L \not\sqsubseteq S^{\textcircled{a}}$ . Hence  $T \sqsubset L$ .

(3) The (images of the) elements  $(Q \times P)/M$ , where  $M$  is a subgroup of  $(Q \times P)$  such that  $q(M)^{\textcircled{a}} \cong L$  (up to conjugation), clearly generate  $\text{Hom}_{RC_p^{\sharp L}}(P, Q)$ . Moreover, the proof of (2) shows that they are linearly independent, since any transitive  $(Q, P)$ -biset  $(Q \times P)/N$  appearing in an

element of the sum  $\sum_{S^\oplus \sqsubset L} RB(Q, S)B(S, P)$  is such that  $q(N)^\oplus \sqsubset L$ . .  $\square$

**7.14. Remark:** If  $G$  is an  $R$ -linear functor from  $RC_p^{\sharp L}$  to the category  $R\text{-Mod}$  of  $R$ -modules, we can extend  $G$  to an  $R$ -linear functor from  $RC_p^L$  to  $R\text{-Mod}$  by setting  $G(P) = \{0\}$  if  $P$  is a finite  $p$ -group such that  $P^\oplus$  is a proper subquotient of  $L$ . Conversely, an  $R$ -linear functor from  $RC_p^L$  to  $R\text{-Mod}$  which vanishes on  $p$ -groups  $P$  such that  $P^\oplus \not\cong L$  can be viewed as an  $R$ -linear functor from  $RC_p^{\sharp L}$  to  $R\text{-Mod}$ . In the sequel, we will freely identify those two types of functors, and consider  $\text{Fun}_R(RC_p^{\sharp L}, R\text{-Mod})$  as the full subcategory of  $\text{Fun}_R(RC_p^L, R\text{-Mod})$  consisting of functors which vanish on  $p$ -groups  $P$  such that  $P^\oplus \not\cong L$ .

**7.15. Theorem:** [  $p \in R^\times$  ] Let  $L$  be an atoric  $p$ -group.

1. If  $F$  is a  $p$ -biset functor over  $R$  such that  $F = \widehat{b}_L F$ , and  $P$  is a finite  $p$ -group such that  $L \not\sqsubseteq P$ , then  $F(P) = \{0\}$ .
2. If  $G$  is an  $R$ -linear functor from  $RC_p^{\sharp L}$  to  $R\text{-Mod}$ , then  $\widehat{b}_L \mathcal{R}_{\mathcal{Y}_L}(G) = \mathcal{R}_{\mathcal{Y}_L}(G)$ .
3. The forgetful functor  $\mathcal{O}_{\mathcal{Y}_L}$  and its right adjoint  $\mathcal{R}_{\mathcal{Y}_L}$  restrict to quasi-inverse equivalences of categories

$$\widehat{b}_L \mathcal{F}_{p,R} \xrightleftharpoons[\mathcal{R}_{\mathcal{Y}_L}]{\mathcal{O}_{\mathcal{Y}_L}} \text{Fun}_R(RC_p^{\sharp L}, R\text{-Mod}) .$$

**Proof :** (1) Since  $\widehat{b}_L F = F$ , then in particular  $F(b_L^P)F(P) = F(P)$ . If  $L \not\sqsubseteq P$ , then there is no minimal section  $(T, S)$  of  $P$  with  $(T/S)^\oplus \cong L$ , thus  $b_L^P = 0$ , and  $F(P) = \{0\}$ .

(2) Let  $G$  be an  $R$ -linear functor from  $RC_p^{\sharp L}$  to  $R\text{-Mod}$ , in other words an  $R$ -linear functor from  $\mathcal{FC}_p^L$  to  $R\text{-Mod}$  which vanishes on  $p$ -groups  $P$  such that  $P^\oplus$  is a proper subquotient of  $L$ . By Theorem 7.9, we have  $\widehat{b}_L^+ \mathcal{R}_{\mathcal{Y}_L}(G) = \mathcal{R}_{\mathcal{Y}_L}(G)$ . If  $H$  is an atoric  $p$ -group which is a proper subquotient of  $L$ , then  $G$  vanishes over any subquotient  $Q$  of  $H$ , since  $Q^\oplus \sqsubseteq H \sqsubset L$  if  $Q \sqsubseteq H$ . In particular  $b_H^P$  acts by 0 on  $\mathcal{R}_{\mathcal{Y}_L}(G)(P)$ , for any finite  $p$ -group  $P$ : indeed  $b_H^P$  is a linear combination of terms of the form  $\text{Indinf}_{X/M}^P \text{Defres}_{X/M}^P$ , where  $(X, M)$  is a section of  $P$  such that  $S \leq M \leq \Phi(T) \leq X \leq T$ , for some section  $(T, S)$  of  $P$  with  $(T/S)^\oplus \cong H$ . For such a section  $(X, M)$  of  $P$ , we have  $(X/M)^\oplus \sqsubseteq (T/S)^\oplus \sqsubseteq H$ , thus  $G$  vanishes on any subquotient of  $X/M$ , so  $\mathcal{R}_{\mathcal{Y}_L}(G)(X/M) = \{0\}$ , hence  $b_H^P = 0$  on  $\mathcal{R}_{\mathcal{Y}_L}(G)(P)$ , as claimed. It follows

that  $\widehat{b}_H \mathcal{R}_{\mathcal{Y}_L}(G) = 0$ , hence  $\widehat{b}_L^+ \mathcal{R}_{\mathcal{Y}_L}(G) = \mathcal{R}_{\mathcal{Y}_L}(G) = \widehat{b}_L \mathcal{R}_{\mathcal{Y}_L}(G)$ .

(3) This is a straightforward consequence of (1) and (2), by Theorem 7.9.  $\square$

The following proposition gives some detail on the structure of the category  $RC_p^{\sharp L}$ :

**7.16. Proposition:** *Let  $p$  be a prime, and  $L$  be an atoric  $p$ -group.*

1. *Let  $P$  be a finite  $p$ -group. Then  $P^{\circ} \cong L$  if and only if there exists an elementary abelian  $p$ -group  $E$  such that  $P \cong E \times L$ .*
2. *Let  $P = E \times L$  and  $Q = F \times L$ , where  $E$  and  $F$  are elementary abelian  $p$ -groups. If  $M \leq (Q \times P)$ , then  $q(M)^{\circ} \cong L$  if and only if*

$$p_{1,2}(M) = p_{2,2}(M) = L \quad \text{and} \quad k_{1,2}(M) = k_{2,2}(M) = \mathbf{1} \quad ,$$

*where  $p_{1,2}$  and  $p_{2,2}$  are the morphisms from  $((H \times L) \times (G \times L))$  to  $L$  defined by  $p_{1,2}((h, x), (g, y)) = x$  and  $p_{2,2}((h, x), (g, y)) = y$ , and*

$$\begin{aligned} k_{1,2}(M) &= \{x \in L \mid ((1, x), (1, 1)) \in M\} \quad , \\ k_{2,2}(M) &= \{x \in L \mid ((1, 1), (1, x)) \in M\} \quad . \end{aligned}$$

**Proof :** (1) This follows from Proposition 6.8.

(2) By Lemma 7.13, the  $R$ -module  $RB(Q, P)$  has a basis consisting of the isomorphism classes of  $(Q, P)$ -bisets of the form  $(Q \times P)/M$ , where  $M$  is a subgroup of  $(Q \times P)$ , up to conjugation, and  $q(M)^{\circ} \cong L$ . If  $M$  is such a subgroup, then  $L \cong (p_1(M)/k_1(M))^{\circ} \subseteq (p_1(M))^{\circ} \subseteq Q^{\circ} \cong L$ , so  $p_1(M)^{\circ} \cong L$ , and similarly  $p_2(M)^{\circ} \cong L$ . By Proposition 6.8  $p_1(M)^{\circ} \cong L$  if and only if  $Ep_1(M) = P$ , which in turn is equivalent to  $p_{1,2}(M) = L$ . Similarly  $p_2(M)^{\circ} \cong L$  if and only if  $p_{2,2}(M) = L$ .

Then  $(p_1(M)/k_1(M))^{\circ} \cong L$  if and only if  $k_1(M) \cap \Phi(p_1(M)) = \mathbf{1}$ , by Proposition 6.6. Moreover  $\Phi(p_1(M)) = \Phi(P)$ , as there exists an elementary abelian subgroup  $E'$  of  $P$  such that  $P = E' \times p_1(M)$ , by Proposition 6.8 again. Since  $\Phi(P) = \mathbf{1} \times \Phi(L)$ , it follows that  $k_1(M) \cap (\mathbf{1} \times \Phi(L)) = \mathbf{1}$ . Now  $N = k_1(L) \cap (\mathbf{1} \times L)$  is a normal subgroup of  $(\mathbf{1} \times L)$  (since  $p_{1,2}(M) = L$ ), which intersect trivially  $(\mathbf{1} \times \Phi(L))$ . Since  $L$  is atoric, by Lemma 6.3, any central element of order  $p$  of  $(\mathbf{1} \times L)$  is contained in  $(\mathbf{1} \times \Phi(L))$ , so  $N$  contains no non trivial central element of  $(\mathbf{1} \times L)$ , hence  $N = \mathbf{1}$ . Thus  $k_1(L) \cap (\mathbf{1} \times L) = \mathbf{1}$ , or equivalently  $k_{1,2}(M) = \mathbf{1}$ . Similarly  $k_{2,2}(M) = \mathbf{1}$ . Hence  $q(M)^{\circ} \cong L$  if and only if  $p_{1,2}(M) = p_{2,2}(M) = L$  and  $k_{1,2}(M) = k_{2,2}(M) = \mathbf{1}$ .  $\square$



## 8. $L$ -enriched bisets

**8.1. Notation:** Let  $G$  and  $H$  be finite groups. If  $U$  is an  $(H, G)$ -biset, and  $u \in U$ , let  $(H, G)_u$  denote the stabilizer of  $u$  in  $(H \times G)$ , i.e.

$$(H, G)_u = \{(h, g) \in (H \times G) \mid hu = ug\} .$$

Let  $H_u = k_1((H, G)_u)$  denote the stabilizer of  $u$  in  $H$ , and  ${}_uG = k_2((H, G)_u)$  denote the stabilizer of  $u$  in  $G$ . Set moreover

$$q(u) = q((H, G)_u) = (H, G)_u / (H_u \times {}_uG) .$$

**8.2. Definition:** Let  $L$  be a finite group. For two finite groups  $G$  and  $H$ , an  $L$ -enriched  $(H, G)$ -biset is a  $(H \times L, G \times L)$ -biset  $U$  such that  $L \sqsubseteq q(u)$ , for any  $u \in U$ . A morphism of  $L$ -enriched  $(H, G)$ -bisets is a morphism of  $(H \times L, G \times L)$ -bisets.

The disjoint union of two  $L$ -enriched  $(H, G)$ -bisets is again an  $L$ -enriched  $(H, G)$ -biset. Let  $B[L](H, G)$  denote the Grothendieck group of finite  $L$ -enriched  $(H, G)$ -bisets for relations given by disjoint union decompositions. The group  $B[L](H, G)$  is called the Burnside group of  $L$ -enriched  $(H, G)$ -bisets.

**8.3. Lemma:** Let  $G, H, L$  be finite groups, and  $U$  be an  $(H \times L, G \times L)$ -biset. Let  $U^{\sharp L}$  denote the set of elements  $u \in U$  such that  $L \sqsubseteq q(u)$ . Then  $U^{\sharp L}$  is the largest sub- $L$ -enriched  $(H, G)$ -biset of  $U$ .

**Proof :** It suffices to show that  $U^{\sharp L}$  is a sub- $(H \times L, G \times L)$ -biset of  $U$ , for then it is clearly the largest sub- $L$ -enriched  $(H, G)$ -biset of  $U$ . And this is straightforward, since for any  $(u, g, h, x, y) \in (U \times G \times H \times L \times L)$ , if  $v = (h, y)u(g, x)^{-1}$ , then

$$(H \times L, G \times L)_v = {}^{((h, y), (g, x))}(H \times L, G \times L)_u ,$$

and this conjugation induces a group isomorphism  $q(v) \cong q(u)$ . □

**8.4. Lemma:** Let  $G, H, L$  be finite groups.

1. Let  $U$  be an  $L$ -enriched  $(H, G)$ -biset. If  $V$  is a sub- $(H \times L, G \times L)$ -biset of  $U$ , then  $V$  is an  $L$ -enriched  $(H, G)$ -biset.

2. The group  $B[L](H, G)$  has a  $\mathbb{Z}$ -basis consisting of the transitive bisets  $((H \times L) \times (G \times L))/M$ , where  $M$  is a subgroup of  $((H \times L) \times (G \times L))$  (up to conjugation) such that  $L \subseteq q(M)$ .

**Proof :** (1) This is straightforward.

(2) It follows from (1) that  $B[L](H, G)$  has a basis consisting of the isomorphism classes of  $L$ -enriched  $(H, G)$ -bisets which are transitive  $(H \times L, G \times L)$ -bisets. These are of the form  $U = ((H \times L) \times (G \times L))/M$ , for some subgroup  $M$  of  $((H \times L) \times (G \times L))$ . Now if  $u$  is the element  $((1, 1), (1, 1))M$  of  $U$ , the group  $(H \times L, G \times L)_u$  is equal to  $M$ , hence  $q(u) \cong q(M)$ .  $\square$

**8.5. Lemma:** Let  $G, H, K, L$  be finite groups.

1. For an  $(H, G)$ -biset  $U$ , endow  $U \times L$  with the  $(H \times L, G \times L)$ -biset structure defined by

$$\forall h \in H, \forall g \in G, \forall x, y, z \in L, \forall u \in U, \quad (h, x)(u, y)(g, z) = (hug, xyz) .$$

Then  $U \times L$  is an  $L$ -enriched  $(H, G)$ -biset.

2. In particular, for any finite group  $G$ , the identity biset of  $G \times L$  is an  $L$ -enriched  $(G, G)$ -biset.
3. If  $U$  is an  $(H, G)$ -biset and  $V$  is a  $(K, H)$ -biset, then there is an isomorphism

$$(V \times L) \times_{(H \times L)} (U \times L) \cong (V \times_H U) \times L$$

of  $L$ -enriched  $(H, G)$ -bisets.

**Proof :** (1) For  $u \in U$  and  $l \in L$ ,

$$(H \times L, G \times L)_{(u, l)} = \{((h, {}^l x), (g, x)) \mid hug = u, l \in L\} \cong (H, G)_u \times L .$$

In particular  $(H \times L)_{(u, l)} = H_u \times \mathbf{1}$  and  ${}_{(u, l)}(G \times L) = {}_u G \times \mathbf{1}$ , and  $q((u, l)) \cong q(u) \times L$  has a (sub)quotient isomorphic to  $L$ .

(2) In particular, if  $H = G$  and  $U$  is the identity  $(G, G)$ -biset, then  $U \times L$  is the identity biset of  $(G \times L)$ .

(3) It is straightforward to check that the maps

$$[(v, x), (u, y)] \in (V \times L) \times_{(H \times L)} (U \times L) \longmapsto ([v, u], xy) \in (V \times_H U) \times L$$

$$[(v, 1), (u, l)] \in (V \times L) \times_{(H \times L)} (U \times L) \longleftarrow ([v, u], l) \in (V \times_H U) \times L$$

are well defined isomorphisms of  $(K \times L, G \times L)$ -bisets, inverse to one another.  $\square$

**8.6. Notation:** Let  $G, H, K, L$  be finite groups. If  $U$  is an  $L$ -enriched  $(H, G)$ -biset and  $V$  is an  $L$ -enriched  $(K, H)$ -biset, let  $V \overset{L}{\times}_H U$  denote the  $L$ -enriched  $(K, G)$ -biset defined by

$$V \overset{L}{\times}_H U = (V \times_{(H \times L)} U)^{\#L}.$$

**8.7. Lemma:** Let  $G, H, J, K, L$  be finite groups.

1. If  $V$  is a  $(K \times L, H \times L)$ -biset and  $U$  is an  $(H \times L, G \times L)$ -biset, then

$$(V \times_{(H \times L)} U)^{\#L} = V^{\#L} \overset{L}{\times}_H U^{\#L}.$$

In particular, if  $V$  and  $U$  are  $L$ -enriched bisets, so is  $V \overset{L}{\times}_H U$ .

2. If  $U$  and  $U'$  are  $L$ -enriched  $(H, G)$ -bisets, if  $V, V'$  are  $L$ -enriched  $(K, H)$ -bisets, then there are isomorphisms

$$\begin{aligned} V \overset{L}{\times}_H (U \sqcup U') &\cong (V \overset{L}{\times}_H U) \sqcup (V \overset{L}{\times}_H U') \\ (V \sqcup V') \overset{L}{\times}_H U &\cong (V \overset{L}{\times}_H U) \sqcup (V' \overset{L}{\times}_H U) \end{aligned}$$

of  $L$ -enriched  $(K, G)$ -bisets.

3. If moreover  $W$  is an  $L$ -enriched  $(J, K)$ -biset, then there is a canonical isomorphism

$$(W \overset{L}{\times}_K V) \overset{L}{\times}_H U \cong W \overset{L}{\times}_K (V \overset{L}{\times}_H U)$$

of  $L$ -enriched  $(J, G)$ -bisets.

**Proof:** (1) Denote by  $[v, u]$  the image in  $V \times_{(H \times L)} U$  of a pair  $(v, u) \in (V \times U)$ . By Lemma 2.3.20 of [7],

$$(K \times L, G \times L)_{[v, u]} = (K \times L, H \times L)_v * (H \times L, G \times L)_u,$$

so by Lemma 2.3.22 of [7], the group  $q([v, u])$  is a subquotient of  $q(v)$  and  $q(u)$ . So if  $[v, u] \in (V \times_{(H \times L)} U)^{\#L}$ , then  $L$  is a subquotient of  $q([v, u])$ , hence it is a subquotient of  $q(v)$  and  $q(u)$ , that is  $v \in V^{\#L}$  and  $u \in U^{\#L}$ . Hence

$$(V \times_{(H \times L)} U)^{\#L} \subseteq (V^{\#L} \times_{(H \times L)} U^{\#L})^{\#L} = V^{\#L} \overset{L}{\times}_H U^{\#L},$$

and the reverse inclusion  $(V^{\#L} \times_{(H \times L)} U^{\#L})^{\#L} \subseteq (V \times_{(H \times L)} U)^{\#L}$  is obvious. Hence  $(V \times_{(H \times L)} U)^{\#L} = V^{\#L} \times_H^L U^{\#L}$ . If  $V$  and  $U$  are  $L$ -enriched bisets, i.e. if  $V = V^{\#L}$  and  $U = U^{\#L}$ , this gives  $(V \times_{(H \times L)} U)^{\#L} = V \times_H^L U$ , so  $V \times_H^L U$  is an  $L$ -enriched biset.

(2) This is straightforward.

(3) With the above notation, there is a canonical isomorphism

$$\alpha : (W \times_{(K \times L)} V) \times_{(H \times L)} U \rightarrow W \times_{(K \times L)} (V \times_{(H \times L)} U)$$

sending  $[[w, v], u]$  to  $[w, [v, u]]$ . Hence

$$\begin{aligned} (W \times_K^L V) \times_H^L U &= ((W \times_K^L V) \times_{(H \times L)} U)^{\#L} \\ &= ((W \times_{(K \times L)} V)^{\#L} \times_{(H \times L)} U)^{\#L} \\ &= ((W \times_{(K \times L)} V) \times_{(H \times L)} U)^{\#L} \quad [\text{by (1)}] \end{aligned}$$

Similarly

$$\begin{aligned} W \times_K^L (V \times_H^L U) &= (W \times_{(K \times L)} (V \times_H^L U))^{\#L} \\ &= (W \times_{(K \times L)} (V \times_{(H \times L)} U)^{\#L})^{\#L} \\ &= (W \times_{(K \times L)} (V \times_{(H \times L)} U))^{\#L} \quad [\text{by (1)}] . \end{aligned}$$

Hence  $\alpha$  induces an isomorphism  $(W \times_K^L V) \times_H^L U \cong W \times_K^L (V \times_H^L U)$ .  $\square$

**8.8. Definition:** Let  $L$  be a finite group, and  $R$  be a commutative ring. The  $L$ -enriched biset category  $\mathcal{RC}[L]$  of finite groups over  $R$  is defined as follows:

- The objects of  $\mathcal{RC}[L]$  are the finite groups.
- For finite groups  $G$  and  $H$ ,

$$\text{Hom}_{\mathcal{RC}[L]}(G, H) = R \otimes_{\mathbb{Z}} B[L](H, G) = RB[L](H, G)$$

is the  $R$ -linear extension of the Burnside group of  $L$ -enriched  $(H, G)$ -bisets.

- The composition in  $\mathcal{RC}[L]$  is the  $R$ -linear extension of the product  $(V, U) \mapsto V \times_H^L U$  defined in 8.6.

- The identity morphism of the group  $G$  is (image in  $RB[L](G, G)$  of) the identity biset of  $G \times L$ , viewed as an  $L$ -enriched  $(G, G)$ -biset.

The category  $RC[L]$  is  $R$ -linear. An  $L$ -enriched biset functor over  $R$  is an  $R$ -linear functor from  $RC[L]$  to  $R\text{-Mod}$ . The category of  $L$ -enriched biset functors over  $R$  is denoted by  $\mathcal{F}_R[L]$ . It is an abelian  $R$ -linear category.

**8.9. Theorem:** Let  $p$  be a prime number, and  $R$  be a commutative ring.

1. If  $L$  is an atomic  $p$ -group, the category  $RC_p^{\sharp L}$  of Definition 7.11 is equivalent to the full subcategory  $R\mathcal{E}l_p[L]$  of  $RC[L]$  consisting of elementary abelian  $p$ -groups.
2. If  $p \in \mathcal{F}^\times$ , the category  $\mathcal{F}_{p,R}$  of  $p$ -biset functors over  $R$  is equivalent to the direct product of the categories  $\text{Fun}_R(R\mathcal{E}l_p[L], R\text{-Mod})$  of  $R$ -linear functors from  $R\mathcal{E}l_p[L]$  to  $R\text{-Mod}$ , for  $L \in [\mathcal{A}t_p]$ .

**Proof :** (1) Let  $E$  be an elementary abelian  $p$ -group. Then  $(E \times L)^\circ \cong L$ , so  $E \times L$  is an object of  $RC_p^{\sharp L}$ . Set  $\mathcal{I}(E) = E \times L$ . If  $E$  and  $F$  are elementary abelian  $p$ -groups, and if  $U$  is a finite  $L$ -enriched  $(F, E)$ -biset, then  $U$  is in particular an  $(F \times L, E \times L)$ -biset, and we can consider its image  $\mathcal{I}(U)$  in the quotient  $\text{Hom}_{RC_p^{\sharp L}}(E \times L, F \times L)$  of  $RB(F \times L, E \times L)$ . This yields a unique  $R$ -linear map  $RB[L](F, E) \rightarrow \text{Hom}_{RC_p^{\sharp L}}(E \times L, F \times L)$ , still denoted by  $\mathcal{I}$ .

We claim that these assignments define a functor  $\mathcal{I}$  from  $R\mathcal{E}l_p[L]$  to  $RC_p^{\sharp L}$ : indeed, the identity  $(E \times L, E \times L)$ -biset is clearly mapped to the identity morphism of  $\mathcal{I}(E)$ . Moreover, if  $G$  is an elementary abelian  $p$ -group, if  $V$  is an  $L$ -enriched  $(G, F)$ -biset and  $U$  is an  $L$ -enriched  $(F, E)$ -biset, it is clear that

$$\mathcal{I}(V \times_{F \times L}^L U) = \mathcal{I}(V) \circ \mathcal{I}(U) ,$$

where the right hand side composition is in the category  $RC_p^{\sharp L}$ : indeed, the transitive bisets  $(Q \times P)/M$  with  $q(M)^\circ \sqsubset L$  appearing in the product  $V \times_{(F \times L)}^L U$  are exactly those vanishing in  $\text{Hom}_{RC_p^{\sharp L}}(\mathcal{I}(E), \mathcal{I}(F))$ , by Lemma 7.13. Hence  $\mathcal{I}$  is an isomorphism

$$\mathcal{I} : RB[L](F, E) \rightarrow \text{Hom}_{RC_p^{\sharp L}}(\mathcal{I}(E), \mathcal{I}(F)) .$$

In other words  $\mathcal{I}$  is a fully faithful functor from  $R\mathcal{E}l_p[L]$  to  $RC_p^{\sharp L}$ . Moreover, by Proposition 6.8, if  $P$  is a finite  $p$ -group with  $P^\circ \cong L$ , there exists an elementary abelian  $p$ -group  $E$  such that  $P$  is isomorphic to  $E \times L$ , hence  $P$  is isomorphic to  $E \times L$  in the category  $RC_p^{\sharp L}$ .

It follows that the functor  $\mathcal{I}$  is fully faithful and essentially surjective, so it is an equivalence of categories.

(2) This is a straightforward consequence of (1), Assertion 5 of Corollary 7.5, and Assertion 3 of Theorem 7.15.  $\square$

## 9. The category $\widehat{b}_L \mathcal{F}_{p,R}$ , for an atoric $p$ -group $L$ ( $p \in R^\times$ )

Let  $L$  be a fixed atoric  $p$ -group. In this section, we give some detail on the structure of the category  $\widehat{b}_L \mathcal{F}_{p,R}$  of  $p$ -biset functors invariant by the idempotent  $\widehat{b}_L$ .

We start by straightforward consequences of Theorem 7.15. For a finite  $p$ -group  $P$ , we denote by  $\Sigma_{\#L}(P)$  the subset of  $\Sigma_L(P)$  consisting of sections  $(X, M)$  of  $P$  such that  $(X/M)^\circ \cong L$ . When  $G$  is an  $R$ -linear functor from  $R\mathcal{C}_p^{\#L}$  to  $R\text{-Mod}$ , we can compute  $\mathcal{R}_{\mathcal{Y}_L}(G)$  at  $P$  by restricting the inverse limit of 7.7 to the subset  $\Sigma_{\#L}(P)$ , i.e. by

$$\mathcal{R}_{\mathcal{Y}_L}(G)(P) = \varprojlim_{(X,M) \in \Sigma_{\#L}(P)} G(X/M) .$$

**9.1. Proposition:** [ $p \in R^\times$ ] *Let  $L$  be an atoric  $p$ -group. If  $F$  is a  $p$ -biset functor in  $\widehat{b}_L \mathcal{F}_{p,R}$ , and  $P$  is a finite  $p$ -group, then*

$$\begin{aligned} F(P) &\cong \varprojlim_{(X,M) \in \Sigma_{\#L}(P)} F(X/M) , \\ &\cong \bigoplus_{\substack{(T,S) \in [\mathcal{M}(P)] \\ (T/S)^\circ \cong L}} \delta_\Phi F(T/S)^{N_P(T,S)/T} . \end{aligned}$$

**Proof :** The isomorphism  $F(P) \cong \varprojlim_{(X,M) \in \Sigma_{\#L}(P)} F(X/M)$  is Assertion 3 of Theorem 7.15. The second isomorphism follows from Theorem 5.4, which implies that for  $(T, S) \in \mathcal{M}(P)$

$$\delta_\Phi F(T/S)^{N_P(T,S)/T} \cong F(\epsilon_{T,S}^P)(F(P)) .$$

Moreover  $F(b_L^P)F(P) = F(P)$  since  $F \in \widehat{b}_L \mathcal{F}_{p,R}$ , and

$$F(\epsilon_{T,S}^P)F(b_L^P) = F(\epsilon_{T,S}^P b_L^P) = 0$$

unless  $(T/S)^\circ \cong L$ . Thus  $\delta_\Phi F(T/S)^{N_P(T,S)/T} = \{0\}$  unless  $(T/S)^\circ \cong L$ , which completes the proof.  $\square$

The decomposition of the category  $\mathcal{F}_{p,R}$  of  $p$ -biset functors stated in Corollary 7.5 leads to the following natural definition:

**9.2. Definition:** [ $p \in R^\times$ ] Let  $F$  be an indecomposable  $p$ -biset functor over  $R$ . There exists a unique atoric  $p$ -group  $L$  (up to isomorphism) such that  $F = \widehat{b}_L F$ . The group  $L$  is called the vertex of  $F$ .

**9.3. Remark:** It follows in particular from this definition that if  $F$  and  $F'$  are indecomposable  $p$ -biset functors over  $R$  with non-isomorphic vertices, then  $\text{Ext}_{\mathcal{F}_{p,R}}^*(F, F') = \{0\}$ .

**9.4. Theorem:** [ $p \in R^\times$ ] Let  $F$  be an indecomposable  $p$ -biset functor over  $R$  and let  $L$  be a vertex of  $F$ . If  $Q$  is a finite  $p$ -group such that  $F(Q) \neq \{0\}$ , but  $F$  vanishes on any proper subquotient of  $Q$ , then  $L \cong Q^\oplus$ .

**Proof :** Let  $Q$  be a finite  $p$ -group such that  $F(Q) \neq \{0\}$  and  $F(Q') = \{0\}$  for any proper subquotient  $Q'$  of  $Q$ . By Proposition 4.6, if  $(T, S)$  is a minimal section of  $Q$ , then

$$\epsilon_{T,S}^Q = \frac{1}{|N_Q(T, S)|} \sum_{\substack{X \leq T, M \leq T \\ S \leq M \leq \Phi(T) \leq X \leq T}} |X| \mu(X, T) \mu_{\leq T}(S, M) \text{Indinf}_{X/M}^Q \circ \text{Defres}_{X/M}^Q .$$

Now if  $X/M$  is a proper subquotient of  $Q$ , i.e. if  $X \neq Q$  or  $M \neq \mathbf{1}$ , then  $F(X/M) = \{0\}$ , and  $F(\text{Indinf}_{X/M}^Q \circ \text{Defres}_{X/M}^Q) = 0$ . Hence  $F(\epsilon_{T,S}^Q) = 0$  unless  $T = Q$  and  $S = \mathbf{1}$ , and moreover

$$F(\epsilon_{Q,\mathbf{1}}^Q) = \frac{1}{|Q|} |Q| \mu(Q, Q) \mu_{\leq Q}(\mathbf{1}, Q) F(\text{Indinf}_{Q/\mathbf{1}}^Q \text{Defres}_{Q/\mathbf{1}}^Q) = \text{Id}_{F(Q)} .$$

If  $\widehat{b}_L F = F$ , then in particular  $F(b_L^Q)$  is equal to the identity map of  $F(Q)$ . This can only occur if the idempotent  $\epsilon_{Q,\mathbf{1}}^Q$  appears in the sum defining  $b_L^Q$ , in other words if  $(Q/\mathbf{1})^\oplus \cong L$ , i.e.  $Q^\oplus \cong L$ . Conversely, if  $Q^\oplus \cong L$ , then  $F(b_L^Q) = F(\epsilon_{Q,\mathbf{1}}^Q) = \text{Id}_{F(Q)} \neq 0$ . It follows that  $\widehat{b}_L F \neq 0$ , hence  $\widehat{b}_L F = F$ , since  $F$  is indecomposable. Hence  $Q^\oplus$  is (isomorphic to) the vertex of  $F$ , as was to be shown.  $\square$

We assume from now on that  $R = k$  is a field. Recall ([7] Chapter 4) that the simple  $p$ -biset functors over  $k$  are indexed by pairs  $(Q, V)$  consisting of a  $p$ -group  $Q$  and a simple  $k\text{Out}(Q)$ -module  $V$ .

**9.5. Corollary:** *Let  $k$  be a field of characteristic different from  $p$ .*

1. *If  $Q$  is a finite  $p$ -group, and  $V$  is a simple  $k\text{Out}(Q)$ -module, then the vertex of the simple  $p$ -biset functor  $S_{Q,V}$  is isomorphic to  $Q^{\textcircled{a}}$ .*
2. *Let  $Q$  (resp.  $Q'$ ) be a finite  $p$ -group, and  $V$  (resp.  $V'$ ) be a simple  $k\text{Out}(Q)$ -module (resp. a simple  $k\text{Out}(Q')$ -module). If  $Q^{\textcircled{a}} \not\cong Q'^{\textcircled{a}}$ , then  $\text{Ext}_{\mathcal{F}_{p,k}}^*(S_{Q,V}, S_{Q',V'}) = \{0\}$ .*

**Proof :** (1) Indeed  $Q$  is a minimal group for  $S_{Q,V}$ , so  $S_{Q,V}(Q) \neq \{0\}$ , but  $S_{Q,V}$  vanishes on any proper subquotient of  $Q$ .

(2) Follows from (1) and Remark 9.3. □

**9.6. Definition:** *Let  $F$  be a  $p$ -biset functor. A functor  $S$  is a subquotient of  $F$  (notation  $S \sqsubseteq F$ ) if there exist subfunctors  $F_2 < F_1 \leq F$  such that  $F_1/F_2 \cong S$ . A composition factor of  $F$  is a simple subquotient of  $F$ .*

**9.7. Lemma:** *Let  $k$  be a field, and  $F$  be a  $p$ -biset functor over  $k$ .*

1. *If  $F$  is a non zero, then  $F$  admits a composition factor.*
2. *If  $\mathcal{S}$  is a family of simple  $p$ -biset functors over  $k$ , there exists a greatest subfunctor of  $F$  all composition factors of which belong to  $\mathcal{S}$ .*

**Proof :** (1) Let  $P$  be a finite  $p$ -group such that  $F(P) \neq \{0\}$ . Then  $F(P)$  is a  $kB(P, P)$ -module. Choose  $m \in F(P) - \{0\}$ , and consider the  $kB(P, P)$ -submodule  $M$  of  $F(P)$  generated by  $m$ . Since  $kB(P, P)$  is finite dimensional over  $k$ , the module  $M$  is also finite dimensional over  $k$ , hence it contains a simple submodule  $V$ . By Proposition 3.1 of [8], there exists a simple  $p$ -biset functor  $S$  such that  $S(P) \cong V$  as  $kB(P, P)$ -module. Then  $S(P)$  is a subquotient of  $F(P)$ , so by Proposition 3.5 of [8], there exists a subquotient of  $F$  isomorphic to  $S$ .

(2) Observe first that if  $M, N$  are subfunctors of  $F$ , then any composition factor of  $M + N$  is a composition factor of  $M$  or a composition factor of  $N$ : indeed, if  $S$  is a composition factor of  $M + N$ , let  $F_2 < F_1 \leq M + N$  with  $S \cong F_2/F_1$ , and consider the images  $F'_1$  and  $F'_2$  of  $F_1$  and  $F_2$ , respectively, in the quotient  $(M + N)/N \cong M/(M \cap N)$ . If  $F'_1 \neq F'_2$ , that is if  $F_1 + N \neq F_2 + N$ , then  $F'_1/F'_2 \cong (F_1 + N)/(F_2 + N) \cong F_1/F_2 \cong S$  is a subquotient of  $(M + N)/N \cong M/(M \cap N)$ , hence  $S$  is a subquotient of  $M$ . Otherwise  $F_1 + N = F_2 + N$ , so  $F_1 = F_2 + (F_1 \cap N)$ , hence  $S \cong F_1/F_2 \cong (F_1 \cap N)/(F_2 \cap N)$



is a subquotient of  $N$ . It follows by induction that any subquotient  $S$  of a finite sum  $\sum_{M \in \mathcal{I}} M$  of subfunctors of  $F$  is a subquotient of some  $M \in \mathcal{I}$ .

The latter also holds when  $\mathcal{I}$  is infinite: let  $\Sigma = \sum_{M \in \mathcal{I}} M$  be an arbitrary sum of subfunctors of  $F$ , and  $S$  be a composition factor of  $\Sigma$ . Let  $F_2 < F_1$  be subfunctors of  $\Sigma$  such that  $S \cong F_1/F_2$ . If  $P$  is a  $p$ -group such that  $S(P) \cong F_1(P)/F_2(P) \neq 0$ , let  $U$  be a finite subset of  $F_1(P)$  such that  $F_1(P)/F_2(P)$  is generated as a  $kB(P, P)$ -module by the images of the elements of  $U$  (such a set exists because  $S(P)$  is finite dimensional over  $k$ , for any  $P$ ). If  $V$  is the  $kB(P, P)$ -submodule of  $F_1(P)$  generated by  $U$ , then  $V$  maps surjectively on the module  $F_1(P)/F_2(P)$ , so there is a  $kB(P, P)$ -submodule  $W$  of  $V$  such that  $V/W \cong S(P)$ . Now since  $U$  is finite, there exists a finite subset  $\mathcal{J}$  of  $\mathcal{I}$  such that  $U \subseteq \sum_{M \in \mathcal{J}} M(P)$ . Setting  $\Sigma_1 = \sum_{M \in \mathcal{J}} M$ , it follows that  $V/W \cong S(P)$  is a subquotient of  $\Sigma_1(P)$ , so by Proposition 3.5 of [8], there exists a subquotient of  $\Sigma_1$  isomorphic to  $S$ . By the observation above  $S$  is a subquotient of some  $M \in \mathcal{J} \subseteq \mathcal{I}$ .

Now let  $\mathcal{I}$  the set of subfunctors  $M$  of  $F$  such that all the composition factors of  $M$  belong to  $\mathcal{S}$ , and  $N = \sum_{M \in \mathcal{I}} M$ . The above discussion shows that  $N \in \mathcal{I}$ , so  $N$  is the greatest element of  $\mathcal{I}$ .  $\square$

**9.8. Theorem:** *Let  $k$  be a field of characteristic different from  $p$ , and  $L$  be an atoric  $p$ -group. Let  $\mathcal{F}_{p,k}[L]$  the full subcategory of  $\mathcal{F}_{p,k}$  consisting of functors whose composition factors all have vertex  $L$ , i.e. are all isomorphic to  $S_{P,V}$ , for some  $p$ -group  $P$  such that  $P^\circ \cong L$ , and some simple  $k\text{Out}(P)$ -module  $V$ .*

1. *If  $F$  is a  $p$ -biset functor, then  $\widehat{b}_L \mathcal{F}_{p,k}$  is the greatest subfunctor of  $F$  which belongs to  $\mathcal{F}_{p,k}[L]$ .*
2. *In particular  $\widehat{b}_L \mathcal{F}_{p,k} = \mathcal{F}_{p,k}[L]$ .*

**Proof :** (1) Let  $F$  be a  $p$ -biset functor over  $k$ , and let  $F_1 = \widehat{b}_L F$ . If  $S$  is a composition factor of  $F_1$ , then  $S = \widehat{b}_L S$ , as  $S$  is a subquotient of  $F_1$ . Hence  $S$  has vertex  $L$ , by Definition 9.2. It follows that  $F_1$  is contained in the greatest subfunctor  $F_2$  of  $F$  which belongs to  $\mathcal{F}_{p,k}[L]$  (such a subfunctor exists by Lemma 9.7).

Conversely, we know that  $F_2 = \bigoplus_{Q \in [\mathcal{A}t_p]} \widehat{b}_Q F_2$ . For  $Q \in [\mathcal{A}t_p]$ , any composition factor  $S$  of  $\widehat{b}_Q F_2$  has vertex  $Q$ , by Definition 9.2. But  $S$  is also a direct summand of  $F_2$ , so  $Q \cong L$ . It follows that if  $Q \not\cong L$ , then  $\widehat{b}_Q F_2$  has no com-

position factor, so  $\widehat{b}_Q F_2 = \{0\}$ , by Lemma 9.7. In other words  $F_2 = \widehat{b}_L F_2$ , hence  $F_2 \leq F_1$ , and  $F_2 = F_1$ , as was to be shown.

(2) Let  $F$  be a  $p$ -biset functor. Then  $F \in \widehat{b}_L \mathcal{F}_{p,k}$  if and only if  $F = \widehat{b}_L F$ , i.e. by (1) if and only if all the composition factors of  $F$  have vertex  $L$ .  $\square$

**9.9. Example: the Burnside functor.** Let  $k$  be a field of characteristic  $q \neq p$  ( $q \geq 0$ ). It was shown in [10] Theorem 8.2 (see also [7] 5.6.9) that the Burnside functor  $kB$  is uniserial, hence indecomposable. As  $kB(\mathbf{1}) \neq 0$ , the vertex of  $kB$  is the trivial group, by Theorem 9.4, thus  $kB$  is an object of  $\widehat{b}_1 \mathcal{F}_{p,k} = \mathcal{F}_{p,k}[\mathbf{1}]$ . It means that all the composition factors of  $kB$  have to be of form  $S_{Q,V}$ , where  $Q^\circledast = \mathbf{1}$ , i.e.  $Q$  is elementary abelian. And indeed by [10] Theorem 8.2, the composition factors of  $kB$  are all of the form  $S_{Q,k}$ , where  $Q$  runs through a specific set of elementary abelian  $p$ -groups which depends on the order of  $p$  modulo  $q$  (suitably interpreted when  $q = 0$ ).

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